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Continuous and Lipschitzian Selections from a Measurable Set-Valued Map

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1. INTRODUCTION

In this paper we construct for a given measurable set-valued map $K: (0, T) \rightsquigarrow \mathbf{R}^n$ with convex and compact values a lower semicontinuous set-valued map $H(\cdot)$ with convex compact values such that $H(t) \subset K(t)$ for almost all $t \in (0, T)$. Moreover, $H(\cdot)$ contains every local continuous selection from $K(\cdot)$, i.e., every continuous function $r: (a, b) \mapsto \mathbf{R}^n$, $(a, b) \subset (0, T)$ such that $r(t) \in K(t)$ for almost all $t \in (a, b)$. Then we construct a set-valued map $L(\cdot)$ with convex compact values such that $L(t) \subset K(t)$ for almost all $t \in (0, T)$, and $L(\cdot)$ contains every Lipschitzian selection from $K(\cdot)$ defined on the previously given open set $P \subset (0, T)$ with the Lipschitz constant less or equal to $k \geqslant 0$. If $L(\cdot)$ is not identically equal to the empty set then it is continuous on P. In both cases we define the maps $H(\cdot)$ and $L(\cdot)$ using the support function.

One motivation for construction of these regularizations of the set-valued map $K(\cdot)$ comes from viability theory (see [1, 2]). If a set-valued map $F: \mathrm{Graph}(K) \rightsquigarrow \mathbf{R}^n$ is given we may regard $K(\cdot)$ as a viability map and we consider the following viability problem

$$\dot{x} \in F(t, x(t)) \tag{1}$$

$$x(t) \in K(t) \qquad \text{for almost all} \quad t \in (0, T). \tag{2}$$

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Since $K(\cdot)$ is not regular enough (for example, with locally compact graph) we cannot use any viability existence theorem for (1), (2) (see [1, 2, 5, 6]). Nevertheless, we may approximate $K(\cdot)$ by $L(\cdot)$. If $L(\cdot)$ has non-empty values we consider differential inclusion (1) with the viability constraint

$$x(t) \in L(t) \qquad \text{for almost all} \quad t \in (0, T). \tag{3}$$

Since the graph of the continuous map $L(\cdot)$ is locally compact, we may use the standard viability argument to check whether (1) has a solution satisfying the viability constraint (3). Such a solution is obviously also a viable solution to (1), (2). Since it is well known that under some continuity assumptions (see [1, p. 91]) the solutions to (1), (2) with $F(x) := f(x, V)$ coincide with the solutions to the following control problem

$$\begin{aligned} &\dot{x}(t) = f(x(t), v(t)) \\ &v(t) \in V \\ &x(t) \in K(t) \qquad \text{for a.a.} \quad t \in (0, T), \end{aligned} \tag{4}$$

we can also get an existence theorem for control problems with the state constraints that depend only measurably on time.

2. Notation and Basic Definitions

$\mathbf{R}^n$ is the Euclidean n-dimensional space; $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbf{R}^n$. By $F: A \rightsquigarrow \mathbf{R}^n$ we denote a set-valued map F, i.e., a map that associates with every $x \in A$ a set $F(x) \subset \mathbf{R}^n$. For $x \notin A$ we set $F(x) = \emptyset$. The domain of a set-valued map $F(\cdot)$ is defined to be $\mathrm{Dom}(F) := \{x \in \mathbf{R}^n \mid F(x) \neq \emptyset\}$. The inverse image of a set M is defined to be $F^{-1}(M) := \{x \in \mathbf{R}^n \mid F(x) \cap M \neq \emptyset\}$. A set-valued map is called essentially bounded if $|F|_\infty := \inf\{\beta > 0 \mid \mu(F^{-1}(\mathbf{R}^n \backslash B(0, \beta))) = 0\} < \infty$, where $B(0, \beta)$ denotes an open ball of radius β and μ denotes Lebesgue measure. We say that $F(\cdot)$ is measurable if $F^{-1}(U)$ is measurable for every open set U, see [3, 4]. We say that $F(\cdot)$ is lower semicontinuous at $x_0 \in \mathrm{Dom}(F)$ if for any $y_0 \in F(x_0)$ and any neighborhood N of y_0, there exists a neighborhood M of x_0 such that for every $x \in M$, $F(x) \cap N \neq \emptyset$, see [1, 3, 4]. $F(\cdot)$ is lower semicontinuous if it is lower semicontinuous at every $x_0 \in \mathrm{Dom}(F)$.

3. Locally Continuous Selections

Definition 1. Let $K: (0, T) \rightsquigarrow \mathbf{R}^n$ be a measurable set-valued map. A continuous map $r: (a, b) \mapsto \mathbf{R}^n$ $(0 \leqslant a < b \leqslant T)$

$$r(t) \in K(t) \qquad \text{for a.a.} \quad t \in (a, b)$$

is called local continuous selection of $K(\cdot)$.

It is convenient to introduce the following definitions:

Definition 2. Let $K: (0, T) \rightsquigarrow \mathbf{R}^n$ be a set-valued map. Then we define $U(\mathrm{Dom}(K))$ to be the maximal open set in $(0, T)$ fulfilling $\mu(U(\mathrm{Dom}(K)) \setminus \mathrm{Dom}(K)) = 0$.

We prove the following theorem:

Theorem 1. *Let $K: (0, T) \rightsquigarrow \mathbf{R}^n$ be a measurable essentially bounded set-valued map with convex and compact values. Then there exists a lower semicontinuous set-valued map $H: (0, T) \rightsquigarrow \mathbf{R}^n$ with convex compact (possibly empty) values, $H(t) \subset K(t)$ for a.a. $t \in (0, T)$ such that for every local continuous selection $r(\cdot)$ from $K(\cdot)$ the following holds:*

$$r(t) \in H(t) \qquad \textit{for } t \textit{ from the domain of } r(\cdot).$$

Remark. If $H(t) = \varnothing$ then there does not exist any local continuous selection from $K(\cdot)$ at the point t. Moreover $\mathrm{Dom}(H)$ may be possibly empty.

To prove Theorem 1 we use the following:

Lemma 1. *Let $P \subset (0, T)$ be an open set and $f: P \mapsto \mathbf{R}$ be a measurable essentially bounded (single valued) map. Then there exists a lower semicontinuous map $g: P \mapsto \mathbf{R}$, such that $g(t) \leqslant f(t)$ for a.a. $t \in P$ and for every continuous map $h: Q \mapsto \mathbf{R}$, where $Q \subset P$ is an open set for which $h(t) \leqslant f(t)$ for a.a. $t \in Q$ holds $h(t) \leqslant g(t)$ for every $t \in Q$.*

Proof. Let $\mathscr{G} := \{p(\cdot) \mid p: P \mapsto \mathbf{R}$ is lower semicontinuous, $p(t) \leqslant f(t)$ for a.a. $t \in P\}$. Since $f(\cdot)$ is essentially bounded, $\mathscr{G}$ is a non-empty set. Let us define

$$g(t) := \sup_{p \in \mathscr{G}} p(t) \qquad \text{for} \quad t \in P.$$

It is easy to see that for every $t \in P$, $g(t) < +\infty$. Indeed, let us suppose $g(t_0) = +\infty$ for some $t_0 \in P$. Since $f(\cdot)$ is essentially bounded there exists a constant $k > 0$ such that

$$f(t) \leqslant k, \qquad \text{for a.a.} \quad t \in (0, T).$$

It follows that there exists $p(\cdot) \in \mathcal{G}$ and $t_0 \in (0, T)$ such that

$$p(t_0) > k.$$

Lower semicontinuity of $p(\cdot)$ implies that the set $\{t \in (0, T) \mid p(t) > k\}$ is non-empty open and therefore

$$p(t) > f(t)$$

on a set of a positive Lebesgue measure.

Since $g(\cdot)$ is supremum of lower semicontinuous functions it is also lower semicontinuous. We prove that

$$g(t) \leqslant f(t) \qquad \text{for a.a.} \quad t \in P.$$

Let us suppose that there exists a set $Z \subset P$ such that $\mu(Z) > 0$ and

$$f(t) < g(t) \qquad \text{for a.a.} \quad t \in Z.$$

Due to the Luzin's theorem for $\mu(Z)/2$ there exists a continuous function $l: P \mapsto \mathbf{R}$ such that

$$\mu(\{t \in P \mid l(t) \neq f(t)\}) < \mu(Z)/2,$$

i.e., there exists a set $A \subset Z$ such that

$$\mu(A) \geqslant \mu(Z)/2$$

and

$$g(t) > l(t) = f(t) \qquad \text{for} \quad t \in A.$$

Let $t_0 \in A$ be a point of density of A, i.e.,

$$\lim_{\eta \to 0_+} \frac{\mu(A \cap (t_0 - \eta, t_0 + \eta))}{2\eta} = 1.$$

It follows that there exists a function $p(\cdot) \in \mathcal{G}$ such that

$$p(t_0) > l(t_0) = f(t_0).$$

Let us denote

$$K := \{t \in P \mid p(t) - l(t) > 0\}.$$

Since $p - l$ is lower semicontinuous, K is nonempty and open. Consequently

$$p(t) > f(t) \qquad \text{on a set of a positive Lebesgue measure.}$$

We got a contradiction with the definition of $g(\cdot)$.

Let $h: Q \mapsto \mathbf{R}$ be a continuous function such that

$$h(t) \leqslant f(t) \qquad \text{for a.a.} \quad t \in Q.$$

Let us define

$$\hat{\mathscr{G}} := \{ p(\cdot) \mid p: Q \mapsto \mathbf{R} \text{ is lower semicontinuous, } p(t) \leqslant f(t) \text{ for a.a. } t \in Q \}$$

and

$$\hat{g}(t) := \sup_{p \in \hat{\mathscr{G}}} p(t) \qquad \text{for} \quad t \in Q.$$

Certainly,

$$\hat{g}(t) = g(t) \qquad \text{for} \quad t \in Q.$$

Since

$$h(\cdot) \in \hat{\mathscr{G}}$$

it follows

$$h(t) \leqslant g(t) \qquad \text{for every} \quad t \in Q.$$

Proof of Theorem 1. Let $e_i \in S$, $i = 1, \ldots$ be a sequence of unit vectors that is dense in the unit sphere S. Let

$$s_i(K(t)) := \sup_{x \in K(t)} \langle x, e_i \rangle, \qquad i = 1, \ldots, \qquad t \in \mathrm{Dom}(K).$$

Let $U := U(\mathrm{Dom}(K))$, see Definition 2. We can consider $s_i(K(\cdot))$ on U. Due to Lemma 1 for the function $s_i(K(\cdot))$ we may construct a maximal (in the sense of Lemma 1) lower semicontinuous function $f_i(\cdot)$ on U. Let

$$\hat{H}(t) := \begin{cases} \{x \in \mathbf{R}^n \mid \langle x, e_i \rangle \leqslant f_i(t), i = 1, \ldots\} & \text{for} \quad t \in U \\ \varnothing & \text{otherwise.} \end{cases}$$

Since the set

$$\hat{D} := \{ t \in (0, T) \mid \hat{H}(t) \neq \varnothing \}$$

is not open in general we define a set-valued map $H: (0, T) \rightsquigarrow \mathbf{R}^n$

$$H(t) := \begin{cases} \hat{H}(t) & \text{for} \quad t \in \mathrm{int}(\hat{D}) \\ \varnothing & \text{for} \quad t \notin \mathrm{int}(\hat{D}). \end{cases}$$

Let

$$D := \mathrm{int}(\hat{D}).$$

Since D is an open set and f_i are lower semicontinuous it follows that $H(\cdot)$ is lower semicontinuous [4, p. 54]. Since

$$f_i(t) \leqslant s_i(K(t)) \qquad \text{for a.a.} \quad t \in U$$

it follows that $H(t) \subset K(t)$, for a.a. $t \in U$, and since $H(t) = \varnothing$ for $t \notin U$ we have

$$H(t) \subset K(t) \qquad \text{for a.a.} \quad t \in (0, T).$$

Let $0 \leqslant a < b \leqslant T$ and

$$r\colon (a, b) \mapsto \mathbf{R}$$

be a local continuous selection from the set-valued map $K(\cdot)$. Since

$$s_i(r(t)) \leqslant s_i(K(t)) \qquad \text{for a.a.} \quad t \in (a, b)$$

and $s_i(r(\cdot))$ is continuous, Lemma 1 implies that

$$s_i(r(t)) \leqslant f_i(t) \qquad \text{for all} \quad t \in (a, b)$$

and consequently

$$r(t) \in H(t) \qquad \text{for all} \quad t \in (a, b).$$

COROLLARY. *Consider the set-valued map $K(\cdot)$ and its lower semicontinuous regularization $H(\cdot)$ as in Theorem 1. Let us denote for every $t \in (0, T)$*

$$M(t) := \{x \in \mathbf{R}^n \mid \textit{there exists a loc. cont. selection } r(\cdot) \textit{ from } K(\cdot),\ r(t) = x\}$$

Then

$$M(t) = H(t).$$

Proof. If $x \in M(t)$ then there exists a local continuous selection $r(\cdot)$ such that $r(t) = x$. Theorem 1 implies $r(t) \in H(t)$.

Conversely, if $x \in H(t_0)$ then the set-valued map

$$\tilde{H}(t) = \begin{cases} H(t) & \text{for} \quad t \neq t_0 \\ \{x\} & \text{for} \quad t = t_0 \end{cases}$$

is lower semicontinuous and due to Michael's selection theorem (see [1, 3, 4]) there exists a continuous selection $r(\cdot)$ such that

$$r(t_0) = x.$$

It follows that $x \in M(t_0)$.

4. Lipschitzian Selections

Definition 3. Let $P \subset \mathbf{R}$, $k \geqslant 0$. By $\mathrm{Lip}(k, P)$ we denote the set of all Lipschitzian maps $g: P \mapsto \mathbf{R}$ with the Lipschitz coefficient less or equal to k.

We prove the following theorem.

Theorem 2. *Let $K: (0, T) \rightsquigarrow \mathbf{R}^n$ be a measurable essentially bounded set-valued map with convex and compact values. Let $k > 0$ and an open set $P \subset U(\mathrm{Dom}(K))$ be given. Then there exists a set-valued map $L: (0, T) \rightsquigarrow \mathbf{R}^n$ with convex compact (possibly empty) values, $L(t) \subset K(t)$ for a.a. $t \in (0, T)$ such that for every Lipschitzian selection $r(\cdot) \in \mathrm{Lip}(k, P)$ from $K(\cdot)$ it holds:*

$$r(t) \in L(t) \qquad \textit{for} \quad t \in P.$$

Moreover if the set-valued map $L(\cdot)$ is not identically equal to $\emptyset$ then it is continuous with nonempty values on P.

The proof of Theorem 2 is analogous to that of Theorem 1.

Lemma 2. *Let $P \subset (0, T)$ be an open set and $f: P \mapsto \mathbf{R}$ be a measurable essentially bounded map. Let $k > 0$ be given. Then there exists a map $g \in \mathrm{Lip}(k, P)$ such that $g(t) \leqslant f(t)$ for a.a. $t \in P$ and for every map $h \in \mathrm{Lip}(k, P)$ for which $h(t) \leqslant f(t)$ for a.a. $t \in P$ holds $h(t) \leqslant g(t)$ for every $t \in P$.*

Proof. Let

$$\mathscr{G} := \{ p(\cdot) \mid p \in \mathrm{Lip}(k, P),\ p(t) \leqslant f(t) \text{ for a.a. } t \in P \}.$$

Let us define

$$g(t) := \sup_{p \in \mathscr{G}} p(t) \qquad \text{for} \quad t \in P.$$

Certainly

$$g(t) < \infty \qquad \text{for} \quad t \in P.$$

We prove that $g(\cdot) \in \mathscr{G}$.

Let $t_1, t_2 \in P$ and $g(t_2) \geqslant g(t_1)$. Let $\varepsilon > 0$ be given. Then there exists $\tilde{p} \in \mathscr{G}$ such that

$$g(t_2) - \tilde{p}(t_2) < \varepsilon.$$

It follows

$$\begin{aligned} 0 \leqslant g(t_2) - g(t_1) &= g(t_2) - \tilde{p}(t_2) + \tilde{p}(t_2) - \tilde{p}(t_1) + \tilde{p}(t_1) - g(t_1) \\ &\leqslant \varepsilon + k\, |t_2 - t_1|. \end{aligned}$$

Since the last formula holds for very $\varepsilon > 0$ it follows

$$g(t_2) - g(t_1) \leqslant k\, |t_2 - t_1|.$$

We proved that $g \in \mathrm{Lip}(k, P)$. To prove that

$$g(t) \leqslant f(t) \qquad \text{for a.a.} \quad t \in P$$

we can use the same argument as in the proof of Theorem 1. Let $h \in \mathrm{Lip}(k, P)$ be such that

$$h(t) \leqslant f(t) \qquad \text{for a.a.} \quad t \in P.$$

From the definition of $g(\cdot)$ it follows that

$$h(t) \leqslant g(t) \qquad \text{for all} \quad t \in P.$$

Proof of Theorem 2. Let $e_i \in S$, $i = 1, \ldots$, be a sequence of unit vectors that is dense in the unit sphere S. Let $U := U(\mathrm{Dom}(K))$ and

$$s_i(K(t)) := \sup_{x \in K(t)} \langle x, e_i \rangle, \qquad i = 1, \ldots, \qquad t \in U.$$

Due to Lemma 2 we may construct functions $f_i(\cdot) \in \mathrm{Lip}(k, P)$ for the functions $s_i(K(\cdot))$. Let

$$\hat{L}(t) := \{x \in \mathbf{R}^n \mid \langle x, e_i \rangle \leqslant f_i(t), i = 1, \ldots\} \qquad \text{for} \quad t \in P.$$

Since the set

$$D := \{t \in P \mid \hat{L}(t) \neq \varnothing\}$$

is not necessarily equal to P we define a set-valued map $L \colon P \rightsquigarrow \mathbf{R}^n$

$$L(t) := \begin{cases} \varnothing & \text{for} \quad t \in (0, T) \text{ if } D \neq P \\ \hat{L}(t) & \text{for} \quad t \in D \text{ if } D = P \\ \varnothing & \text{for} \quad t \in (0, T) \setminus D \text{ if } D = P. \end{cases}$$

Since D is an open set and f_i are continuous, it follows that $L(\cdot)$ is continuous on D (see [4, p. 54]). Since

$$f_i(t) \leqslant s_i(K(t)) \qquad \text{for a.a.} \quad t \in P$$

it follows that $L(t) \subset K(t)$, for a.a. $t \in P$. Let $r \in \mathrm{Lip}(k, P)$ be a Lipschitzian selection from the set-valued map $K(\cdot)$. Since

$$s_i(r(t)) \leqslant s_i(K(t)) \qquad \text{for a.a.} \quad t \in P$$

and $s_i(r(\cdot)) \in \mathrm{Lip}(k, P)$ is Lipschitzian it follows from Lemma 2 that

$$s_i(r(t)) \leqslant f_i(t) \qquad \text{for all} \quad t \in P$$

and consequently

$$r(t) \in L(t) \qquad \text{for all} \quad t \in P.$$

REFERENCES

1. J.-P. AUBIN AND A. CELLINA, "Differential Inclusions," Springer, Berlin, 1984.
2. J.-P. AUBIN, Viability theory, to appear.
3. J.-P. AUBIN AND H. FRANKOWSKA, "Set-Valued Analysis," Birkhäuser, Boston, 1990.
4. C. CASTAING AND M. VALADIER, Convex analysis and measurable multifunctions, *in* "Lecture Notes in Mathematics," Vol. 580, Springer, Berlin, 1977.
5. K. DEIMLING, Multivalued differential equations on closed sets II, *Differential Integral Equations*, to appear.
6. G. HADDAD, Monotone trajectories of differential inclusions and functional differential inclusions with memory, *Israel J. Math.* **39** (1981), 83–100.