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ABSOLUTELY CONTINUOUS SELECTIONS FROM ABSOLUTELY CONTINUOUS SET VALUED MAP

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INTRODUCTION

In this paper we prove that the barycentric selection from an absolutely continuous set valued map $F: (0, T) \rightsquigarrow \mathbf{R}^n$ with nonempty convex values is absolutely continuous. Moreover we prove using the barycentric selection that under certain conditions for every $x_0 \in F(t_0)$ there exists an absolutely continuous selection $f(\cdot)$ from a set valued map $F(\cdot)$ such that $f(t_0) = x_0$.

The existence of an absolutely continuous selection plays an important role in the viability theory (see [1]) if the viability map $K(\cdot)$ depends only measurably on time. Then the necessary condition for the existence of a viable solution is the existence of an absolutely continuous selection from $K(\cdot)$.

NOTATION

$\mathbf{R}^n$ is the Euclidian n-dimensional space; $d(x, y)$ is the Euclidian distance from x to y. $B(x, M)$ denotes the open ball of radius M about x and $B := B(0, 1)$. S denotes the unit sphere. If A, B are subsets of $\mathbf{R}^n$, $d(x, A) := \inf \{d(x, y) \mid y \in A\}$, $\delta(A, B) := := \sup \{d(x, B) \mid x \in A\}$ denotes the separation of A from B and $d^*(A, B) := := \sup (\delta(A, B), \delta(B, A))$ is Hasudorff distance of the sets A and B. For $x, y \in \mathbf{R}^n$, $\langle x, y \rangle$ denotes the scalar product.. Let $A \subset \mathbf{R}^n$, $A \neq \emptyset$, $e \in S$ then $\sigma_A(e) := \sup_{a \in A} \langle a, e \rangle$ is the support function of the set A. By $\mathrm{ri}(A)$ we denote the relative interior of the set A.

MAIN RESULTS

Definition 1. Let $F: (0, T) \rightsquigarrow \mathbf{R}^n$ be a set valued map with convex and compact values. We say that F is an *absolutely continuous map* if the following condition is fulfilled

$$\forall \varepsilon > 0\,, \quad \exists \delta > 0 \quad \text{such that for every system of intervals}$$

$$[t_1, \tau_1], \dots, [t_m, \tau_m]\,, \quad (0 \leqq t_1 \leqq \tau_1 \leqq \dots \leqq t_m \leqq \tau_m \leqq T)$$

the following holds

$$\sum_{j=1}^{m}(\tau_j - t_j) < \delta \Rightarrow \max\left(\sum_{j=1}^{m}\mu_n((F(t_j)+B)\setminus(F(\tau_j)+B))\right.,$$
$$\left.\sum_{j=1}^{m}\mu_n((F(\tau_j)+B)\setminus(F(t_j)+B))\right) < \varepsilon\,,$$

where μ_n denotes n-dimensional Lebesgue measure.

Let $A \subset \mathbf{R}^n$ be a convex compact set with nonempty interior. Then we define (see [1])

$$b(A) := \frac{1}{\mu_n(A)}\int_A x\,\mathrm{d}\mu_n\,.$$

Theorem 1. *Let $F: (0, T) \rightsquigarrow \mathbf{R}^n$ be an absolutely continuous set valued map with nonempty convex and compact values. Let $F(\cdot)$ be bounded, i.e. there exists $M > 0$ such that*

$$\forall t \in (0, T)\,, \quad F(t) \subset M\,.\,B\,.$$

Then the map $f: (0, T) \mapsto \mathbf{R}^n$

$$f(t) := \mathrm{b}(F(t) + B)$$

is an absolutely continuous selection from $F(\cdot)$.

To prove this theorem we use the following lemma.

Lemma 1 (see Aubin and Cellina, 1984, p. 78). *Let $A \subset \mathbf{R}^n$ be a convex and compact set and $A_1 := A + B$. Then $\mathrm{b}(A_1) \in A$.*

Proof of theorem 1. Let

$$\Phi(t) := F(t) + B\,.$$

Let $\varepsilon > 0$. Since $F(\cdot)$ is an absolutely continuous set valued map there exists $\delta > 0$ such that for every system of intervals

$$[t_1, \tau_1], \ldots, [t_m, \tau_m]\,, \quad (0 \leqq t_1 \leqq \tau_1 \leqq \ldots \leqq t_m \leqq \tau_m \leqq T)$$

holds

$$\sum_{j=1}^{m}(\tau_j - t_j) < \delta \Rightarrow \max\left(\sum_{j=1}^{m}\mu_n(\Phi(t_j)\setminus\Phi(\tau_j))\right.,$$
$$\left.\sum_{j=1}^{m}\mu_n(\Phi(\tau_j)\setminus\Phi(t_j))\right) < \varepsilon\,\mu_n(B)/(4(M+1))\,.$$

It follows that

$$\sum_{i=1}^{m}\|f(t_i) - f(\tau_i)\| = \sum_{i=1}^{m}\left\|\frac{1}{\mu_n(\Phi(t_i))}\int_{\Phi(t_i)} x\,\mathrm{d}\mu_n - \frac{1}{\mu_n(\Phi(\tau_i))}\int_{\Phi(\tau_i)} x\,\mathrm{d}\mu_n\right\| \leqq$$
$$\leqq \sum_{i=1}^{m}\left(\left\|\left(\frac{1}{\mu_n(\Phi(t_i))} - \frac{1}{\mu_n(\Phi(\tau_i))}\right)\int_{\Phi(t_i)\cap\Phi(\tau_i)} x\,\mathrm{d}\mu_n\right\| + \right.$$
$$\left. + \left\|\frac{1}{\mu_n(\Phi(t_i))}\int_{\Phi(t_i)\setminus\Phi(\tau_i)} x\,\mathrm{d}\mu_n - \frac{1}{\mu_n(\Phi(\tau_i))}\int_{\Phi(\tau_i)\setminus\Phi(t_i)} x\,\mathrm{d}\mu_n\right\|\right).$$

Using lemma 1 and boundedness of the map $F(\cdot)$ we get

$$\mu_n(\Phi(t_i)) \geqq \mu_n(B), \quad \mu_n(\Phi(\tau_i)) \geqq \mu_n(B), \quad i = 1, \ldots, m,$$

$$\left\| \left(\frac{1}{\mu_n(\Phi(t_i))} - \frac{1}{\mu_n(\Phi(\tau_i))} \right) \int_{\Phi(t_i) \cap \Phi(\tau_i)} x \, \mathrm{d}\mu_n \right\| \leqq$$

$$\leqq |\mu_n(\Phi(\tau_i)) - \mu_n(\Phi(t_i))| \, (M + 1)/\mu_n(B),$$

$$\left\| \frac{1}{\mu_n(\Phi(t_i))} \int_{\Phi(t_i) \setminus \Phi(\tau_i)} x \, \mathrm{d}\mu_n - \frac{1}{\mu_n(\Phi(\tau_i))} \int_{\Phi(\tau_i) \setminus \Phi(t_i)} x \, \mathrm{d}\mu_n \right\| \leqq$$

$$\leqq (\mu_n(\Phi(t_i) \setminus \Phi(\tau_i)) + \mu_n(\Phi(\tau_i) \setminus \Phi(t_i))) \, (M + 1)^2/\mu_n(B).$$

Since $F(\cdot)$ is an absolutely continuous map

$$\sum_{i=1}^{m} |\mu_n(\Phi(t_i)) - \mu_n(\Phi(\tau_i))| =$$

$$= \sum_{i=1}^{m} |\mu_n(\Phi(t_i) \setminus \Phi(\tau_i)) - \mu_n(\Phi(\tau_i) \setminus \Phi(t_i))| < \mu_n(B) \, \varepsilon/(2(M + 1)^2).$$

Using these estimates we get

$$\sum_{i=1}^{m} \|f(t_i) - f(\tau_i)\| < \varepsilon.$$

We proved that $f(\cdot)$ is absolutely continuous on the interval $(0, T)$. □

Lemma 2. *Let $M > 0$. Then there exists $k > 0$ such that for every two nonempty convex and compact sets, $C, D \subset \mathbf{R}^n$ such that $C, D \subset M \, . \, B$ holds*

$$kd^*(C, D) \geqq \max\left[\mu_n((C + B) \setminus (D + B)), \mu_n((D + B) \setminus (C + B))\right].$$

Proof. We prove that there exists $k_1 > 0$ such that

$$k_1 \delta(C, D) = k_1 \delta(C + B, \ D + B) \geqq \mu_n((C + B) \setminus (D + B)).$$

There exists $k_1 > 0$ (see [1], p. 80) such that

$$\mu_n((C + B) \setminus (D + B)) \leqq \mu_n(B(D + B, \delta(C + B, D + B)) -$$

$$- \mu_n(D + B) \leqq k_1 \delta(C + B, D + B).$$

Similarly we prove that there exists $k_2 > 0$ such that

$$k_2 \delta(D, C) = k_2 \delta(D + B, C + B) \geqq \mu_n((D + B) \setminus (C + B)).$$

Let

$$k := \max(k_1, k_2).$$

Then

$$kd^*(C, D) \geqq \max\left[\mu_n((C + B) \setminus (D + B)), \ \mu_n((D + B) \setminus (C + B))\right].$$

□

The following definition was used by Kikuchi and Tomita, [3].

Definition 2. Let $F: (0, T) \rightsquigarrow \mathbf{R}^n$ be a set valued map with nonempty compact values. We say that F is *d^*-absolutely continuous* if for $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for every system of intervals

$$[t_1, \tau_1], \ldots, [t_m, \tau_m], \quad (0 \leqq t_1 \leqq \tau_1 \leqq \ldots \leqq t_m \leqq \tau_m \leqq T)$$

the following holds

$$\sum_{j=1}^{m} (\tau_j - t_j) < \delta \Rightarrow \sum_{j=1}^{m} d^*(F(t_j), F(\tau_j)) < \varepsilon .$$

From lemma 2 follows:

Lemma 3. *Let $F\colon (0, T) \rightsquigarrow \mathbf{R}^n$ be a bounded, d^*-absolutely continuous set valued map with nonempty convex compact values. Then $F(\cdot)$ is an absolutely continuous map.*

Lemma 4. *Let $F\colon (0, T) \rightsquigarrow \mathbf{R}^n$ be a set valued map with nonempty convex and compact values and $h\colon (0, T) \mapsto \mathbf{R}^n$ be an absolutely continuous function such that*

$$\forall e \in S, \quad \forall t, \tau \in (0, T), \quad \sigma_{F(t)}(e) - \sigma_{F(\tau)}(e) \leqq |h(t) - h(\tau)| .$$

Then $F(\cdot)$ is d^*-absolutely continuous set valued map.

Proof. Using the minimax theorem (see [2]) we get

$$\delta(F(t) + B, F(\tau) + B) = \sup_{e \in S} \sup_{y \in F(t)+B} \inf_{x \in F(\tau)+B} \langle e, y - x \rangle =$$

$$= \sup_{e \in S} \left(\sup_{y \in F(t)+B} \langle e, y \rangle - \sup_{x \in F(\tau)+B} \langle e, x \rangle \right) = \sup_{e \in S} \left(\sigma_{F(t)+B}(e) - \sigma_{F(\tau)+B}(e) \right) =$$

$$= \sup_{e \in S} \left(\sigma_{F(t)}(e) - \sigma_{F(\tau)}(e) \right) = \delta(F(t), F(\tau)) .$$

It follows that

$$\delta(F(t), F(\tau)) \leqq |h(t) - h(\tau)|, \quad \delta(F(\tau), F(t)) \leqq |h(\tau) - h(t)|,$$

i.e.

$$d^*(F(t), F(\tau)) \leqq |h(t) - h(\tau)| .$$

Since $h(\cdot)$ is an absolutely continuous function then $F(\cdot)$ is d^*-absolutely continuous set valued map. □

Theorem 2. *Let $H\colon (0, T) \rightsquigarrow \mathbf{R}^n$ be a bounded set valued map with nonempty convex and compact values and let $t_0 \in (0, T)$, $x_0 \in H(t_0)$. Let $h\colon (0, T) \mapsto \mathbf{R}^n$ be an absolutely continuous function such that*

$$\forall e \in S, \quad \forall t, \tau \in (0, T), \quad \sigma_{H(t)}(e) - \sigma_{H(\tau)}(e) \leqq |h(t) - h(\tau)| .$$

Then there exists $\delta > 0$ and an absolutely continuous selection $r\colon [t_0, t_0 + \delta) \mapsto \mathbf{R}^n$ from $H(\cdot)$ such that

$$r(t_0) = x_0 .$$

To prove theorem 2 we will use the following definition and lemma.

Definition 3. Let L, K be linear subspaces in $\mathbf{R}^n$. Let $\Pi_L(\cdot)$ denote the projection of the best approximation on the set L. We define

$$\alpha(L, K) := \sup\{1 - \|\Pi_L(x)\| \mid x \in K, \|x\| = 1\}.$$

Lemma 5. *Let $H \subset B(0, R)$, $(R > 0)$ be a convex compact set, L be a linear subspace of $\mathbf{R}^n$, $L \subset \operatorname{aff}(H) - \operatorname{aff}(H)$ ($\operatorname{aff}(H)$ denotes the affine hull of the set H, see [4]) and there exists $x_0 \in \mathbf{R}^n$ and $\delta > 0$ such that*

$$B(x_0, \delta) \cap (L + x_0) \subset H.$$

Let K be a linear subspace in $\mathbf{R}^n$ such that $K + L = \mathbf{R}^n$ and $K \cap L = \{0\}$. Let $L_0 := N_K(0)$ ($N_K(0)$ denotes the normal cone to K at 0, see [1]) and $\alpha(L, L_0) < 1$. Then there exists a constant $r > 0$ such that

$$\sigma_{H \cap K}(e) = \inf\{\sigma_H(e') + \sigma_K(e'') \mid e' + e'' = e, \|e'\| + \|e''\| \leqq r\}, \forall e \in S,$$

where r depends only on n, α, R, δ.

To prove lemma 5 we use the following two lemmas.

Lemma 6. *Let L, L_0 be linear subspaces in $\mathbf{R}^n$ and let $\dim(L_0) = \dim(L)$, $\alpha := \alpha(L, L_0) < 1$. Then the projection map $\Pi_L \colon L_0 \mapsto L$ has an inverse $\Pi^{-1} \colon L \mapsto L_0$ and*

$$\|\Pi_L^{-1}\| := \sup\{\|\Pi_L^{-1}(y)\| \mid y \in L, \|y\| = 1\} \leqq 1/(1 - \alpha).$$

Proof. Let Q be an subspace in $\mathbf{R}^n$ orthogonal to L such that $Q + L = \mathbf{R}^n$. If $\dim(L_0 \cap Q) \geqq 1$, then there exists $q \in Q \cap L_0$, $\|q\| = 1$. Since $\Pi_L(q) = 0$ it follows that $\alpha = 1$. This contradicts with the assumption $\alpha < 1$. We proved that $L_0 \cap Q = \{0\}$. For given $y \in L$ since $\dim L + \dim Q = \dim L_0 + \dim Q = n$ there exists exactly one $x \in L_0$ such that $\Pi_L(x) = y$. From the definition of α follows that $\|\Pi_L(x)\| \geqq (1 - \alpha)\|x\|$ and therefore $\|\Pi_L^{-1}\| \leqq 1/(1 - \alpha)$. □

Lemma 7. *Let $L_0 := \{x \in \mathbf{R}^n \mid x_1 = \ldots = x_k = 0\}$, $K := \{x \in \mathbf{R}^n \mid x_{k+1} = \ldots = x_n = 0\}$, L be a linear subspace in $\mathbf{R}^n$, $\dim(L) = \dim(L_0)$ and $\alpha := \alpha(L, L_0) < 1$, $c > 0$. Let*

$$Z := \{x \in \mathbf{R}^n \mid \sum_{i=1}^{k} x_i^2 \leqq c^2, \|\Pi_L(x)\| \leqq c\}.$$

Then for every $y \in Z$ the following holds

$$\|y\| \leqq \frac{3c(n + 2 - \alpha)}{(1 - \alpha)^2}.$$

Proof. Let $y \in Z$. Due to lemma 6 there exists only one $x^y \in L_0$ such that

$$\Pi_L(x^y) = \Pi_L(y) \quad \text{and} \quad \|x^y\| \leqq \frac{c}{1 - \alpha}.$$

Let Q be an subspace in $\mathbf{R}^n$ orthogonal to L such that $Q + L = \mathbf{R}^n$. Let $a^1, \ldots, a^k$

be an orthonormal basis of the space Q such that

$$a^1 = \frac{x^y - y}{\|x^y - y\|}\,.$$

For every $x \in \mathbf{R}^n$

$$\Pi_L(x) = x - \sum_{s=1}^{k} \langle x, a^s \rangle\, a^s\,.$$

Let

$$\delta := \frac{\alpha - 1}{\alpha - n - 2}\,.$$

We prove

$$\sum_{i=1}^{k} (a_i^1)^2 \geqq \delta^2\,.$$

Let us suppose that

$$\sum_{i=1}^{k} (a_i^1)^2 < \delta^2\,. \tag{1}$$

Let

$$b_i := a_i^1\,, \quad i = 1, \ldots, k$$

$$b_i := 0\,, \quad i = k + 1, \ldots, n$$

and

$$\hat{x} := a^1 - b \in L_0\,.$$

It follows

$$\|b\| < \delta\,, \quad \|\hat{x}\| > 1 - \delta\,,$$

$$\Pi_L(\hat{x}) = -b + \sum_{s=1}^{k} \langle b, a^s \rangle\, a^s\,.$$

Since

$$\|\Pi_L(\hat{x})\| \leqq (n + 1)\,\delta$$

then

$$\alpha \geqq 1 - \frac{\|\Pi_L(\hat{x})\|}{\|\hat{x}\|} > 1 - \frac{(n + 1)\,\delta}{1 - \delta}\,.$$

Since

$$1 - \frac{(n + 1)\,\delta}{1 - \delta} = \alpha$$

we get the contradiction with the assumption (1). It follows

$$\sum_{i=1}^{k} (a_i^1)^2 \geqq \delta^2\,.$$

Since

$$y = x^y + t a^1$$

then

$$\sum_{i=1}^{k} (x_i^y + ta_i^1)^2 \leqq c^2$$

and consequently

$$|t| \leqq \frac{\sqrt{(\sum_{i=1}^k (x_i^y)^2)} + c}{\sqrt{(\sum_{i=1}^k (a_i^1)^2)}} \leqq \frac{2c}{(1-\alpha)\,\delta}.$$

It follows

$$\|y\| \leqq \|x^y\| + \frac{2c}{(1-\alpha)\,\delta} \leqq \frac{3c}{(1-\alpha)\,\delta} = \frac{3c(n+2-\alpha)}{(1-\alpha)^2}. \qquad \square$$

Proof of lemma 5. By translation and unitary transformation, we can achieve $x_0 = 0$ and $K = \{x \in \mathbf{R}^n \mid x_{k+1} = \ldots = x_n = 0\}$ where $n - k = \dim(L)$. Let

$$L_0 := \{x \in \mathbf{R}^n \mid x_1 = \ldots = x_k = 0\}.$$

From the assumptions we get

$$\sigma_H(e) \geqq \sigma_{L \cap H}(e) = \sigma_{L \cap H}(\Pi_L(e)) \geqq \delta \|\Pi_L(e)\|.$$

Moreover (see [4] and $e \in S$)

$$R \geqq \sigma_{H \cap K}(e) = \inf\{\sigma_H(e') + \sigma_K(e'') \mid e' + e'' = e\} \geqq$$
$$\geqq \inf\{\delta \|\Pi_L(e')\| \mid e' + e'' = e,\ e'' \in L_0\}.$$

We get

$$\|\Pi_L(e')\| \leqq R/\delta \quad \text{whenever } e - e' \in L_0.$$

Since

$$\sum_{i=1}^{k} e_i''^2 = 0$$

then

$$\sum_{i=1}^{k} e_i'^2 \leqq 1.$$

From lemma 7 follows

$$\sigma_{H \cap K}(e) = \inf\{\sigma_H(e') + \sigma_K(e'') \mid e' + e'' = e,\ \|e'\| + \|e''\| \leqq r\},$$
$$\forall e \in S,$$

where

$$r := \frac{6R(n+2-\alpha)}{\delta(1-\alpha)^2} + 1. \qquad \square$$

Proof of theorem 2. A) Let $\dim(\operatorname{aff} H(t_0)) = 0$, i.e. $H(t_0) = \{x_0\}$. Then for barycentric selection holds

$$\mathfrak{b}(H(t_0) + B) = x_0.$$

Therefore we may define due to theorem 1 an absolutely continuous selection

$$r(t) := \mathfrak{b}(H(t) + B).$$

B) Let $x_0 \in \mathrm{ri}(H(t_0))$, $\dim \operatorname{aff} H(t_0) = n - k \geqq 1$ and K be a linear subspace such that $K = N_{\operatorname{aff}(H(t_0))}(x_0)$. By translation and unitary transformation we can achieve $x_0 = 0$ and

$$K := \{x \in \mathbf{R}^n \mid x_{k+1} = \ldots = x_n = 0\}.$$

From Carathéodory theorem (see [1]) follows that there exist points $a_i \in H(t_0)$, $i = 1, \ldots, n - k + 1$ such that

$$x_0 = \sum_{i=1}^{n-k+1} \lambda_i a_i, \quad \sum_{i=1}^{n-k+1} \lambda_i = 1, \quad \lambda_i > 0.$$

Let $\eta_1, \eta_2 > 0$ be such that

$$B(x_0, \eta_1) \cap \operatorname{aff}(b_1, \ldots, b_{n-k+1}) \subset$$
$$\subset \{x \in \mathbf{R}^n \mid x = \sum_{i=1}^{n-k+1} \mu_i b_i, \ \sum_{i=1}^{n-k+1} \mu_i = 1, \ \mu_i \geqq 0\}$$

for every $\|b_i - a_i\| < \eta_2$, $i = 1, \ldots, n - k + 1$.

Since $H(\cdot)$ is a continuous map then for $\eta_2 > 0$ there exists $\delta_1 > 0$ such that

$$H(t) \cap B(a_i, \eta_2/2) \neq \emptyset \quad \text{for} \quad |t - t_0| < \delta_1.$$

Let $b_i(t) \in H(t) \cap B(a_i, \eta_2/2)$ and

$$L_0 := \operatorname{aff}\{a_1, \ldots, a_{n-k+1}\} = \{x \in \mathbf{R}^n \mid x_1 = \ldots = x_k = 0\},$$
$$L(t) := \operatorname{aff}\{b_1(t), \ldots, b_{n-k+1}(t)\}.$$

Let $R > 0$ be such that $H(t) \subset B(0, R)$ for $t \in [t_0, t_0 + \delta)$ and

$$G(t) := H(t) \cap K.$$

We find $\delta_2 > 0$ such that $\alpha(L(t), L_0) < 1/2$ for $t \in [t_0, t_0 + \delta_2)$, $\delta := \min(\delta_1, \delta_2)$. For $t \in [t_0, t_0 + \delta)$ are fulfilled the assumptions of lemma 5, where x_0 stands for $x(t) \in L(t) \cap K$, L stands for $L(t)$. Therefore there exists $r > 0$ such that

$$\sigma_{G(t)}(e) = \inf\{\sigma_{H(t)}(e') \mid e' + e'' = e, \ e'' \in L_0, \ \|e'\| + \|e''\| \leqq r\}.$$

We prove that

$$\sigma_{G(t)}(e) - \sigma_{G(\tau)}(e) \leqq r|h(t) - h(\tau)|, \quad \forall t, \ \tau \in [t_0, t_0 + \delta), \quad \forall e \in S.$$

Since $\sigma_{K(t)}(\cdot)$ is lower semicontinuous function (see [4]) it follows that for every $\tau \in [t_0, t_0 + \delta)$ and every $e \in S$ there exists $\tilde{e} \in \mathbf{R}^n$, $\|\tilde{e}\| \leqq r$ such that $e - \tilde{e} \in L_0$

$$\sigma_{H(\tau)}(\tilde{e}) = \sigma_{G(\tau)}(e) = \inf\{\sigma_{H(\tau)}(e') \mid e' + e'' = e, \ e'' \in L_0, \ \|e'\| + \|e''\| \leqq r\}.$$

It follows

$$\sigma_{G(t)}(e) - \sigma_{G(\tau)}(e) \leqq \sigma_{H(t)}(\tilde{e}/\|\tilde{e}\|)\,\|\tilde{e}\| - \sigma_{H(\tau)}(\tilde{e}/\|\tilde{e}\|)\,\|\tilde{e}\| \leqq$$
$$\leqq r|h(t) - h(\tau)|.$$

From lemma 4 and the first part of this proof it follows that there exists an absolutely continuous selection $r(\cdot)\colon [t_0, t_0 + \delta) \mapsto \mathbf{R}^n$ from the set valued map $G(\cdot)$,

$$r(t_0) = x_0.$$

C) Let $x_0 \in \mathrm{bd}(H(t_0))$. Let us suppose that $\dim \operatorname{aff} H(t_0) \geqq 1$. Take $y \in \mathrm{ri}(H(t_0))$ and let $\lambda_n \in \mathbf{R}$, $n = 1, \dots$ be a decreasing sequence such that $1 \geqq \lambda_1$, $\lambda_n \to 0$. Let

$$y_n := x_0(1 - \lambda_n) + y\lambda_n .$$

Since $y_n \in \mathrm{ri}(H(t_0))$ there exists an absolutely continuous selection $x_n(\cdot)$ defined on the interval $t_0 \leqq t < t_0 + \eta_n$ such that

$$x_n(t_0) = y_n .$$

There exists $\delta_n < \min(1/n, \eta_n)$, $n = 1, \dots$ such that $\delta_{n+1} \leqq \delta_n$,

$$\operatorname*{var}_{[t_0, t_0+\delta_n]} x_n < 1/n^2 ,$$

$$\operatorname*{var}_{[t_0, t_0+\delta_{n+1}]} x_{n+1} < 1/n^2 .$$

Let

$$x(t, \lambda) := \frac{\lambda - \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} x_n(t) + \frac{\lambda_n - \lambda}{\lambda_n - \lambda_{n+1}} x_{n+1}(t)$$

$$\text{for } \lambda_{n+1} \leqq \lambda \leqq \lambda_n , \quad t_0 < t \leqq t_0 + \delta_{n+1}$$

$$x(t_0, 0) := x_0 .$$

Since $x_n(\cdot)$, $n = 1, \dots$ are absolutely continuous and $H(\cdot)$ has convex values there exists an increasing continuously differentiable function $\hat{\delta} \in C^1[0, 1]$, $\hat{\delta}(\lambda_k) \leqq \delta_{k+1}$, $\hat{\delta}(\lambda) > 0$ for $1 \geqq \lambda > 0$, $\hat{\delta}(0) = 0$ such that

$$x(t, \lambda) \in H(t) \quad \text{for} \quad t_0 \leqq t \leqq t_0 + \hat{\delta}(\lambda) .$$

Let the function $\hat{\lambda}(t)$ be the inverse function for $t_0 + \hat{\delta}(\lambda)$. We prove that

$$\hat{x}(t) := x(t, \hat{\lambda}(t))$$

is absolutely continuous on the interval $[t_0, t_0 + \hat{\delta}(1)]$.

We prove that for every $\varepsilon > 0$ there exists $k \in N$ such that $\hat{x}(\cdot)$ has variation on the interval $[t_0, t_0 + \hat{\delta}(\lambda_k)]$ less then ε. Let $\varepsilon > 0$. We choose $k \in N$ such that

$$\lambda_k \|x_0 - y\| + 4 \sum_{n \geqq k} 1/n^2 < \varepsilon/2 .$$

There exist points $t_i \in (t_0, t_0 + \hat{\delta}(\lambda_k)]$, $i = 1, \dots, M + 1$ such that $t_i < t_{i+1}$

$$\Big| \operatorname*{var}_{[t_0, t_0+\delta(\lambda_k)]} \hat{x} - \sum_{i=1}^{M} \|\hat{x}(t_{i+1}) - \hat{x}(t_i)\| \Big| < \varepsilon/2 .$$

We add the points $t_0 + \hat{\delta}(\lambda_s)$ for $s \geqq k$ if $t_1 < t_0 + \hat{\delta}(\lambda_s)$ to the points t_i. Let $\lambda_{n+1} \leqq \leqq \hat{\lambda}(t_i) \leqq \hat{\lambda}(t_{i+1}) \leqq \lambda_n$. Then

$$\hat{x}(t_{i+1}) - \hat{x}(t_i) = -(\hat{\lambda}(t_{i+1}) - \hat{\lambda}(t_i))(x_0 - y) +$$

$$+ \frac{\hat{\lambda}(t_i) - \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} (x_n(t_{i+1}) - x_n(t_i)) +$$

$$+ \frac{\lambda_n - \hat{\lambda}(t_{i+1})}{\lambda_n - \lambda_{n+1}} (x_{n+1}(t_{i+1}) - x_{n+1}(t_i)) +$$

$$+ \frac{\hat{\lambda}(t_{i+1}) - \hat{\lambda}(t_i)}{\lambda_n - \lambda_{n+1}} (x_n(t_i) - y_n - x_{n+1}(t_i) + y_{n+1}).$$

It follows that

$$\sum_{i=1}^{M} \|\hat{x}(t_{i+1}) - \hat{x}(t_i)\| \leqq \sum_{i=1}^{M} |\hat{\lambda}(t_{i+1}) - \hat{\lambda}(t_i)| \, \|x_0 - y\| +$$

$$+ \sum_{n \geqq k} \sum_{\{i \in \{1,\dots,M\} \mid \lambda_{n+1} \leqq \hat{\lambda}(t_i) \leqq \hat{\lambda}(t_{i+1}) \leqq \lambda_n\}} \|x_n(t_{i+1}) - x_n(t_i)\| +$$

$$+ \sum_{n \geqq k} \sum_{\{i \in \{1,\dots,M\} \mid \lambda_{n+1} \leqq \hat{\lambda}(t_i) \leqq \hat{\lambda}(t_{i+1}) \leqq \lambda_n\}} \|x_{n+1}(t_{i+1}) - x_{n+1}(t_i)\| +$$

$$+ \sum_{n \geqq k} \sum_{\{i \in \{1,\dots,M\} \mid \lambda_{n+1} \leqq \hat{\lambda}(t_i) \leqq \hat{\lambda}(t_{i+1}) \leqq \lambda_n\}} \frac{\hat{\lambda}(t_{i+1}) - \hat{\lambda}(t_i)}{\lambda_n - \lambda_{n+1}} \Big(\operatorname*{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_n +$$

$$+ \operatorname*{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_{n+1} \Big) \leqq \lambda_k \|x_0 - y\| + \sum_{n \geqq k} \Big(\operatorname*{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_n +$$

$$+ \operatorname*{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_{n+1} + \operatorname*{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_n + \operatorname*{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_{n+1} \Big) \leqq$$

$$\leqq \lambda_k \|x_0 - y\| + 4 \sum_{n \geqq k} 1/n^2 < \varepsilon/2 .$$

We proved that

$$\operatorname*{var}_{[t_0, t_0 + \delta(\lambda_k)]} \hat{x} < \varepsilon .$$

We prove that $\hat{x}(\cdot)$ is an absolutely continuous function on the interval $[t_0 + \hat{\delta}(\lambda_k), t_0 + \hat{\delta}(1)]$. Since $x_n(\cdot)$ is an absolutely continuous function on the interval $[0, \delta_n]$ then for $\forall \varepsilon > 0$. $\exists \eta > 0$ such that for every system of intervals

$$[t_1, \tau_1], \dots, [t_m, \tau_m], \quad (t_0 + \hat{\delta}(\lambda_k) \leqq t_1 \leqq \tau_1 \leqq \dots \leqq t_m \leqq \tau_m \leqq \delta_n)$$

the following holds

$$\sum_{j=1}^{m} (\tau_j - t_j) < \eta \Rightarrow \sum_{j=1}^{m} \|x_n(t_j) - x_n(\tau_j)\| < \varepsilon/(4k), \; n \leqq k .$$

It follows that for every system of intervals

$$[t_1, \tau_1], \dots, [t_m, \tau_m], \quad (t_0 + \hat{\delta}(\lambda_k) \leqq t_1 \leqq \tau_1 \leqq \dots \leqq t_m \leqq \tau_m \leqq \hat{\delta}(1))$$

holds

$$\sum_{j=1}^{m} (\tau_j - t_j) < \eta \Rightarrow \sum_{n \leqq k} \sum_{\{i \in \{1,\dots,m\} \mid \lambda_{n+1} \leqq \hat{\lambda}(t_i) \leqq \hat{\lambda}(\tau_i) \leqq \lambda_n\}} \|x_n(t_i) - x_n(\tau_i)\| < \varepsilon/4 .$$

We proved that $\hat{x}(\cdot)$ is an absolutely continuous on the interval $[t_0 + \hat{\delta}(\lambda_k), t_0 + \hat{\delta}(1)]$ and since $\operatorname{var}_{[t_0, t_0 + \delta(\lambda_k)]} \to 0$ for $k \to \infty$, then it is absolutely continuous on the interval $[t_0, t_0 + \hat{\delta}(1)]$. □

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