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TECHNICAL NOTE

On the Intersection of Contingent Cones

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Abstract. In this paper, we give a new condition that ensures the equality $T_K(x) \cap T_L(x) = T_{K \cap L}(x)$ for convex closed sets K, L. This condition, which is given in terms of support functions of the sets K, L, generalizes, in a Hilbert space, the usual condition $0 \in \mathrm{int}(K-L)$.

Key Words. Contingent cone, Clarke tangent cone, calculus on contingent cones, support function.

1. Introduction

Different tangent cones, like the Bouligand contingent cone, the Clarke tangent cone, etc, play an important role in nonsmooth analysis, control theory, viability theory, etc (see Refs. 1–4). In the case of convex sets, these cones coincide and are called the tangent cone. Calculus on contingent (or tangent) cones may be found, for example, in Refs. 1–4. In general, the contingent cone to the intersection of two sets is not equal to the intersection of the corresponding contingent cones. Some attempt has been made to find a condition under which the equality would be ensured. In Refs. 1–3, it was proved that, if $A \in \mathscr{L}(X, Y)$ is a linear continuous map, with X, Y Banach spaces, if $L \subset X$, $K \subset Y$ are convex closed subsets, and if

$$0 \in \mathrm{int}(K - A(L)), \tag{1}$$

then for every $x \in A^{-1}(K) \cap L$,

$$A^{-1}(T_K(A(x))) \cap T_L(x) = T_{A^{-1}(K) \cap L}(x). \tag{2}$$

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Since the proof of this theorem was based on the Robinson–Ursescu theorem (see Ref. 1, Theorem 1.3.1), condition (1) was essential. On the other hand, it is not difficult to see that there are cases in which (1) is not satisfied and (2) is still valid. Let $X = Y = \mathbf{R}^2$, K be a triangle with vertices in points $(0, 0)$, $(1, 1)$, $(-1, 1)$. Let $L = -K$ and $A(x) := x$. Clearly, $0 \notin \mathrm{int}(K - L)$, yet

$$T_K(0) \cap T_L(0) = T_{K \cap L}(0).$$

In this note, we assume that X, Y are Hilbert spaces. Using support functions of convex subsets of a Hilbert space, we prove that the following condition implies Eq. (2): there exists $c > 0$ such that, for every $e \in X^*$,

$$\sigma_{L \cap A^{-1}(K)}(e) = \inf\{\sigma_L(e - A^*(e')) + \sigma_K(e') \mid e' \in Y^*, \|e'\|_{Y^*} \le c\|e\|_{X^*}\}. \tag{3}$$

We show that condition (3) is satisfied in the above mentioned example. Moreover, we prove that, if L is bounded, then (1) implies (3).

2. Main Result

Let X be a Hilbert space and $K \subset X$. Let

$$\sigma_K(e) := \sup_{x \in K} \langle e, x \rangle, \qquad e \in X^*,$$

denote the support function of the set K (see Refs. 1–4, 6). Let $x \in K$. Then, the Bouligand contingent cone is defined to be

$$T_K(x) := \left\{ v \in X \mid \liminf_{h \downarrow 0_+} [\mathrm{dist}(K, x + hv)/h] = 0 \right\},$$

and the Clarke tangent cone is

$$C_K(x) := \left\{ v \in X \mid \liminf_{\substack{h \downarrow 0_+ \\ x' \xrightarrow{K} x}} [\mathrm{dist}(K, x' + hv)/h] = 0 \right\},$$

where $x' \xrightarrow{K} x$ denotes the convergence in the set K; see Refs. 1–4.

Let us recall that, for a convex set K, these two cones coincide (Refs. 2–3).

Theorem 2.1. Let X and Y be Hilbert spaces; let $L \subset X$, $K \subset Y$ be nonempty closed convex sets and $A \in \mathscr{L}(X, Y)$ be a continuous linear map. By $A^* \in \mathscr{L}(Y^*, X^*)$, we denote the transpose of A. Let there exist a constant $c > 0$ such that, for every $e \in X^*$,

$$\sigma_{L \cap A^{-1}(K)}(e) = \inf\{\sigma_L(e - A^*(e')) + \sigma_K(e') \mid e' \in Y^*, \|e'\|_{Y^*} \le c\|e\|_{X^*}\}. \tag{4}$$

Let $x \in L \cap A^{-1}(K)$. Then,

$$T_{L \cap A^{-1}(K)}(x) = T_L(x) \cap A^{-1}(T_K(A(x))).$$

Proof. Since the inclusion

$$T_{A^{-1}(K) \cap L}(x) \subset T_{A^{-1}(K)}(x) \cap T_L(x) \subset A^{-1}(T_K(A(x))) \cap T_L(x)$$

is always true (see Ref. 1, p. 225), we have to prove that

$$A^{-1}(T_K(A(x))) \cap T_L(x) \subset T_{A^{-1}(K) \cap L}(x).$$

Let

$$v \in T_L(x) \cap A^{-1}(T_K(A(x))).$$

From the definition of the Clarke tangent cone, it follows that, for every sequence $x_n \to x$, $x_n \in L \cap A^{-1}(K)$, and every sequence $h_n \downarrow 0_+$, there exist sequences $v_n \to A(v)$, $u_n \to v$, such that

$$x_n + h_n u_n \in L, \qquad A(x_n) + h_n v_n \in K.$$

Since L and K are convex sets, it follows that, for every $e \in X^*$, $e' \in Y^*$, and every $n \in \mathrm{N}$,

$$\sigma_L(e) \geq \langle x_n + h_n u_n, e \rangle, \qquad \sigma_K(e') \geq \langle A(x_n) + h_n v_n, e' \rangle.$$

Let

$$e \in b(L \cap A^{-1}(K)),$$

where b stands for the barrier cone,

$$b(L) := \{e \in X^* \mid \sigma_L(e) < \infty\}, \qquad L \subset X,$$

(see Ref. 1), i.e.,

$$i := \sigma_{L \cap A^{-1}(K)}(e) < \infty.$$

From (4), it follows that there exist e'_j, $j = 1, \ldots,$ such that, for $\|e'_j\|_{Y^*} \leq c\|e\|_{X^*}$,

$$\sigma_L(e - A^*(e'_j)) + \sigma_K(e'_j) \to i.$$

Consequently, for every $\epsilon > 0$, there exists $j_o > 0$ such that, for every $j \geq j_o$ and every $n \in \mathrm{N}$,

$$\begin{aligned}\sigma_{L \cap A^{-1}(K)}(e) &\geq \sigma_L(e - A^*(e'_j)) + \sigma_K(e'_j) - \epsilon \\ &\geq \langle x_n + h_n u_n, e - A^*(e'_j) \rangle + \langle A(x_n) + h_n v_n, e'_j \rangle - \epsilon \\ &= \langle x_n + h_n u_n, e \rangle - h_n \langle A(u_n) - v_n, e'_j \rangle - \epsilon.\end{aligned}$$

Since

$$\|e_j'\|_{Y^*}\leq c\|e\|_{X^*}, \qquad \text{for every } j=1,\ldots,$$

it follows that

$$\langle x_n+h_nu_n, e\rangle \leq \sigma_{L\cap A^{-1}(K)}(e)+h_nc\|e\|_{X^*}\|A(u_n)-v_n\|_Y+\epsilon.$$

Since the last expression holds for every $\epsilon>0$ and every $e\in b(L\cap A^{-1}(K))$, it follows that

$$x_n+h_nu_n\in \bar{B}_X(L\cap A^{-1}(K), h_nc\|A(u_n)-v_n\|_Y),$$

where $\bar{B}_X(x, r)$ denotes the closed ball in X centered at x with the radius r. It follows that, for every $n\in\mathrm{N}$, we can find $y_n\in L\cap A^{-1}(K)$, such that

$$\|x_n+h_nu_n-y_n\|_X\leq h_nc\|A(u_n)-v_n\|_Y.$$

Let

$$u_n':=(y_n-x_n)/h_n, \qquad h_n':=h_n.$$

Then, we have

$$\|u_n'-u_n\|_X\leq c\|A(u_n)-v_n\|_Y, \tag{5a}$$

$$x_n+h_n'u_n'\in L\cap A^{-1}(K). \tag{5b}$$

From (5), it follows that

$$u_n' \to v.$$

Hence,

$$v\in T_{A^{-1}(K)\cap L}(x).$$

□

Corollary 2.1. Let $L, K\subset X$ be two nonempty closed convex subsets of a Hilbert space X, and let there exist a constant $c>0$ such that, for every $e\in X^*$,

$$\sigma_{L\cap K}(e)=\inf\{\sigma_L(e-e')+\sigma_K(e') \mid e'\in X^*, \|e'\|_{X^*}\leq c\|e\|_{X^*}\}. \tag{6}$$

Let $x\in L\cap K$. Then,

$$T_{L\cap K}(x)=T_L(x)\cap T_K(x).$$

Remark 2.1. Let $X=\mathbf{R}^2$. We identify X with its dual X^*. It is easy to see that (6) is satisfied in the example given in the introduction. Since

$$\bar{B}_X(0, 1)\subset(N_K(0)\cap\bar{B}_X(0, 1))+(N_L(0)\cap\bar{B}_X(0, 1))$$

[$N_K(0)$ denotes the normal cone to the set K at 0, see Refs. 1–2], it follows that, for every $e \in X$, there exist $e_1 \in (N_K(0) \cap \bar{B}_X(0, 1))$ and $e_2 \in (N_L(0) \cap \bar{B}_X(0, 1))$ such that

$$e = \|e\|_X e_1 + \|e\|_X e_2.$$

Let

$$\bar{e}' := \|e\|_X e_1.$$

Since $\bar{e}' \in N_K(0)$ and $e - \bar{e}' \in N_L(0)$, it follows that

$$\inf\{\sigma_K(e') + \sigma_L(e - e') \mid e' \in X,\ \|e'\|_X \leq \|e\|_X\} \leq \sigma_K(\bar{e}') + \sigma_L(e - \bar{e}') = 0.$$

From the convexity of the support function $\sigma_{K \cap L}$, it follows that

$$\sigma_{K \cap L}(e) \leq \sigma_{K \cap L}(\bar{e}') + \sigma_{K \cap L}(e - \bar{e}') \leq \sigma_K(\bar{e}') + \sigma_L(e - \bar{e}').$$

Since $\sigma_{K \cap L}(e) = 0$ for every $e \in X$, it follows that

$$0 = \sigma_{K \cap L}(e) = \inf\{\sigma_K(e') + \sigma_L(e - e') \mid e' \in X,\ \|e'\|_X \leq \|e\|_X\}.$$

Therefore, condition (6) is satisfied, and consequently

$$T_{L \cap K}(0) = T_L(0) \cap T_K(0).$$

Proposition 2.1. Let X and Y be Hilbert spaces; let $L \subset X$, $K \subset Y$ be nonempty closed convex sets, $L \subset \bar{B}_X(0, p)$, for some $p > 0$; and let $A \in \mathscr{L}(X, Y)$ be a continuous linear map. Let

$$0 \in \operatorname{int}(K - A(L)).$$

Then, there exists $c > 0$ such that (4) is fulfilled.

To prove Proposition 2.1, we will use the following lemma.

Lemma 2.1. Under the assumptions of Proposition 2.1, for every $l > p$, there exists $c > 0$ such that, for every $e \in X^*$,

$$K_l(e) := \{e' \in Y^* \mid \sigma_K(e') + \sigma_L(e - A^*(e')) \leq l\|e\|_{X^*}\} \subset c\|e\|_{X^*}\bar{B}_{Y^*}(0, 1). \tag{7}$$

Proof. Let $z_0 \in Y$. Since $0 \in \operatorname{int}(K - A(L))$, it follows that there exists $r > 0$ such that

$$\bar{B}_Y(0, r) \subset K - A(L).$$

Consequently, there exist $x_o \in L$, $y_o \in K$, such that

$$rz_o/\|z_o\|_Y = y_o - A(x_o).$$

Let $e \in X^*$ be fixed. Let, for every $e' \in Y^*$,

$$U(e') := \sigma_K(e') + \sigma_L(e - A^*(e')).$$

For the conjugate function $U^*: Y \mapsto \mathbf{R}$ (see Ref. 6), the following estimate holds true:

$$\begin{aligned} U^*(rz_o/\|z_o\|_Y) &= \sup_{e' \in Y^*} (\langle rz_o/\|z_o\|_Y, e'\rangle - \sigma_K(e') - \sigma_L(e - A^*(e'))) \\ &\leq \sup_{e' \in Y^*} (\langle y_o - A(x_o), e'\rangle - \langle y_o, e'\rangle - \langle x_o, e - A^*(e')\rangle) \\ &= -\langle x_o, e\rangle \leq p\|e\|_{X^*}. \end{aligned}$$

The Fenchel inequality (see Ref. 6, p. 29) implies that

$$\sup_{e' \in K_l(e)} \langle e', rz_o/\|z_o\|_Y\rangle \leq l\|e\|_{X^*} + U^*(rz_o/\|z_o\|_Y) \leq (l+p)\|e\|_{X^*}.$$

We proved that, for every $e \in X^*$ and every $z \in Y$,

$$\sigma_{K_l(e)}(z) \leq \|z\|_Y/r\|e\|_{X^*}(l+p).$$

Hence,

$$\forall e \in X^*, K_l(e) \subset \|e\|_{X^*}(l+p)/r\bar{B}_{Y^*}(0, 1).$$

Condition (7) is satisfied if we set

$$c := (l+p)/r.$$ □

Proof of Proposition 2.1. Let $l > p$. Since

$$A^{-1}(K) \cap L \subset \bar{B}_X(0, p),$$

it follows that

$$\sigma_{A^{-1}(K) \cap L}(e) \leq p\|e\|_{X^*}, \qquad \text{for every } e \in X^*.$$

Since

$$\sigma_{L \cap A^{-1}(K)}(e) = \inf\{\sigma_L(e - A^*(e')) + \sigma_K(e') \mid e' \in Y^*\}$$

(see Ref. 2, p. 31), it follows that

$$K_l(e) \neq \emptyset, \qquad \text{for every } e \in X^*.$$

Since $e' \mapsto \sigma_K(e') + \sigma_L(e - A^*(e'))$ is a lower semicontinuous convex function, it follows that $K_l(e)$ is a nonempty convex closed set. From Lemma 2.1 and Ref. 2, p. 31, it follows that there exists

$$e'_1 \in Y^*, \qquad \|e'_1\|_{Y^*} \leq c\|e\|_{X^*},$$

such that

$$\inf_{e' \in Y^*} \sigma_K(e') + \sigma_L(e - A^*(e')) = \inf_{e' \in K_l(e)} \sigma_K(e') + \sigma_L(e - A^*(e'))$$
$$= \sigma_K(e'_1) + \sigma_L(e - A^*(e'_1)),$$

and therefore (4) is satisfied. □

If X is an n-dimensional Euclidean space (with Euclidean norm denoted by $\|.\|$) identified with its dual, a condition ensuring (6) was given in Ref. 5. We recall it here.

Definition 2.1. Let L, K be linear subspaces in $\mathbf{R}^n$. Let Π_L denote the projection of the best approximation on the set L. We define

$$\alpha(L, K) := \sup\{1 - \|\Pi_L(x)\| \,|\, x \in K, \|x\| = 1\}.$$

Proposition 2.2. Let $H \subset B(0, R)$, $(R > 0)$ be a convex compact set; let L be a linear subspace of $\mathbf{R}^n$, $L \subset \mathrm{aff}(H) - \mathrm{aff}(H)$ [$\mathrm{aff}(H)$ denotes the affine hull of the set H, see Ref. 6]; and let there exist $x_o \in \mathbf{R}^n$ and $\delta > 0$ such that

$$B(x_o, \delta) \cap (L + x_o) \subset H.$$

Let K be a linear subspace in $\mathbf{R}^n$ such that $K + L = \mathbf{R}^n$ and $K \cap L = \{0\}$. Let $L_o = N_K(0)$ and $\alpha(L, L_o) < 1$. Then, there exists a constant $c > 0$ such that

$$\sigma_{H \cap K}(e) = \inf\{\sigma_H(e') + \sigma_K(e'') \,|\, e' + e'' = e, \|e'\| + \|e''\| \leq c\|e\|\}, \qquad \forall e \in \mathbf{R}^n,$$

where c depends only on α, R, δ.

References

1. AUBIN, J. P., and CELLINA, A., *Differential Inclusions*, Springer-Verlag, Berlin, Germany, 1984.
2. AUBIN, J. P., and EKELAND, I., *Applied Nonlinear Analysis*, John Wiley, New York, New York, 1984.

3. AUBIN, J. P., and FRANKOWSKA, H., *Set-Valued Analysis*, Birkhäuser, Boston, Massachusetts, 1990.
4. CLARKE, F. H., *Optimization and Nonsmooth Analysis*, John Wiley, New York, New York, 1983.
5. KŘIVAN, V., and VRKOČ, I., *Absolutely Continuous Selections from an Absolutely Continuous Set-Valued Map*, Czechoslovak Mathematical Journal, Vol. 40, No. 3, pp. 503–513, 1990.
6. ROCKAFELLAR, R. T., *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.