JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **155**, 131–139 (1991)

Perturbation of Viability Problem

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Submitted by George Leitmann

Received February 21, 1989

1. INTRODUCTION

Let us consider a control problem

$$\begin{aligned} \dot{x}(t) &= f(x(t), v(t)) \\ v(t) &\in V(x(t)), \end{aligned} \tag{1}$$

where the set-valued map $V(\cdot)$ denotes a priori feedbacks, see [2–4]. Moreover, let a set K that defines the state constraints be given, i.e.,

$$x(t) \in K. \tag{2}$$

Let us assume that control problem (1) with state constraints (2) has a solution. Let $\tilde{K}$ denote a perturbation of the set K, $\tilde{f}(\cdot, \cdot)$ and $\tilde{V}(\cdot)$ denote perturbations of $f(\cdot, \cdot)$ and $V(\cdot)$. The question is whether for such perturbed control problem there exists a solution, i.e., a control such that the corresponding trajectory will satisfy the perturbed constraints $x(t) \in \tilde{K}$. A non-stochastic approach to robustness in control problems with disturbances, perturbations, etc., has been advocated by G. Leitmann and his follow workers [10–13] and A. Kurzhanski [7–9]. We use here a viability approach to touch these issues. Let us recall that control problem (1) is in fact a differential inclusion

$$\dot{x}(t) \in F(x(t)), \tag{3}$$

where $F(x) := f(x, V(x))$. We can study at least two invariance properties of the set K with respect to (3). The first one, called invariance, consists in the fact that all solutions for (3) starting from any initial point $x_0 \in K$ should stay in K. The second one, called also viability [2, 3], requires that for any initial point $x_0 \in K$ there exists at least one trajectory to (3)

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satisfying (2). Such a trajectory is called viable. There is no doubt that from the control theory point of view this approach is more adequate than invariance, since it means that there exists at least one control $v(\cdot)$ for control problem (1) such that the corresponding trajectory stays in K.

Moreover, viability theory [2, 3] plays an important role in many "soft" sciences like economics, biology, etc. Quite often we do not know exactly either the viability set K or the right hand side of differential inclusion $F(\cdot)$. This leads us to study the perturbed viability problem

$$\begin{aligned} \dot{x}(t) &\in F(x(t)) + \xi(x(t)) \\ x(t) &\in K(u) \\ x(0) &= x_0, \end{aligned} \tag{4}$$

where the set-valued map $K(\cdot)$ denotes perturbations acting on the set K of constraints and the set-valued map $\xi(\cdot)$ denotes the perturbation of $F(\cdot)$. The question is under which conditions the perturbed problem (4) has a viable solution. There are several consequences of this problem both in biology (and in the other soft sciences as well) that mainly motivated this paper and in the theory of control systems with state constraints or systems under uncertainty. For example, every biological system must satisfy certain constraints that define the viability set. These constraints can be disturbed unpredictably by external forces in the course of time. Therefore the system must be able to survive at least within some small range of the perturbations. The measure of the robustness for the system under study may be given by the range of the perturbations that do not destroy the viability property of the system.

Let us recall that the main viability theorem [2, 3] says that for upper semicontinuous set-valued map $F(\cdot)$ with convex and compact values and for closed K the necessary and sufficient condition (in finite dimension) for existence of a viable solution to (3) is the condition

$$\forall x \in K, \qquad F(x) \cap T_K(x) \neq \emptyset,$$

where $T_K(x)$ denotes the contingent cone.

2. Notation and Basic Definitions

$\mathbf{R}^n$ is the Euclidean n-dimensional space; $d(x, y)$ is the Euclidean distance from x to y. $B(x, M)$ denotes the open ball of radius M about x and $B := B(0, 1)$. S denotes the unit sphere. If A, C are subsets of $\mathbf{R}^n$, $A + C := \{a + c \mid a \in A,\ c \in C\}$, $A - C := \{a - c \mid a \in A,\ c \in C\}$, $d(x, A) := \inf\{d(x, y) \mid y \in A\}$, $\delta(A, C) := \sup\{d(x, C) \mid x \in A\}$ denotes the separation of A from C, and $d^*(A, C) := \sup(\delta(A, C), \delta(C, A))$ is the Hausdorff distance of the sets A and B. By $\sigma_A\colon S \mapsto \mathbf{R}^n$ we denote the support function of the set A, i.e., $\sigma_A(e) := \sup_{a \in A} \langle a, e \rangle$ where $\langle \cdot, \cdot \rangle$ stands for scalar product. Let $x \in A$ then $T_A(x) := \bigcap_{\varepsilon > 0} \bigcap_{\alpha > 0} \bigcup_{h < \alpha} (1/h \cdot (A - x) + \varepsilon \cdot B)$ denotes the contingent (or Bouligand) cone, see [2–4]. If $K_i \subset \mathbf{R}^n$, $i = 1, \ldots$, then by $\liminf_{n \to \infty} K_n$ we denote the Kuratowski lower limit of the sequence K_i, i.e., $\liminf_{n \to \infty} K_n := \bigcap_{\varepsilon > 0} \bigcup_{N > 0} \bigcap_{n \geq N} B(K_n, \varepsilon)$, see [5, 6, 13, 14].

3. Main Theorem

Theorem 1. *Let X be a Banach space and let $K\colon X \rightsquigarrow \mathbf{R}^n$ be a lower semicontinuous set-valued map with nonempty convex compact values. Let $u_0 \in X$ be given and $F\colon \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ be a bounded set-valued map with nonempty values which is lower semicontinuous on the boundary of $K(u_0)$. Let*

$$F(x) \cap \operatorname{int}(T_{K(u_0)}(x)) \neq \emptyset, \qquad \forall x \in K(u_0).$$

Then there exists $\varepsilon > 0$ and $\eta > 0$ such that for every $u \in X$, $\|u - u_0\| \leq \varepsilon$ holds $K(u) \cap B(K(u_0), \eta) \neq \emptyset$, and

$$\forall x \in K(u) \cap B(K(u_0), \eta), \qquad F(x) \cap T_{K(u)}(x) \neq \emptyset. \tag{5}$$

Moreover, if the set-valued map $K(\cdot)$ is continuous then there exists $\varepsilon > 0$ such that for every $u \in X$, $\|u - u_0\| \leq \varepsilon$

$$\forall x \in K(u), \qquad F(x) \cap T_{K(u)}(x) \neq \emptyset. \tag{6}$$

Remark. Let $K(\cdot)$ be continuous. From viability theory [2, 3] follows, that in the case when $F(\cdot)$ is a bounded upper semicontinuous set-valued map with nonempty convex and closed values, that is, continuous on the boundary of $K(u_0)$, condition (6) ensures the existence of a viable solution to the problem

$$\begin{aligned} &\dot{x}(t) \in F(x(t)) \\ &x(t) \in K(u) \\ &x(0) = x_0 \in K(u) \end{aligned} \tag{7}$$

for every $u \in X$, $\|u - u_0\| \leq \varepsilon$.

Remark. Let $K \subset \mathbf{R}^n$ and

$$T_K(x) \cap F(x) \neq \emptyset, \qquad \forall x \in K.$$

Let

$$\xi \colon K \leadsto \mathbf{R}^n$$

such that

$$\forall x \in K, \qquad \xi(x) \cap \left(\bigcap_{y \in K} (T_K(y) - F(y)) \right) \neq \emptyset. \tag{8}$$

Then

$$T_K(x) \cap (F(x) + \xi(x)) \neq \emptyset, \qquad \forall x \in K.$$

For $F(\cdot)$, $\xi(\cdot)$ upper semicontinuous with nonempty convex compact values condition (8) ensures existence of a viable solution to the problem

$$\begin{aligned} &\dot{x}(t) \in F(x(t)) + \xi(x(t)) \\ &x(t) \in K \\ &x(0) = x_0 \in K. \end{aligned} \tag{9}$$

To prove Theorem 1 we need the following two simple lemmas.

LEMMA 1. *Let X be a Banach space and let $K \colon X \leadsto \mathbf{R}^n$ be a lower semicontinuous set-valued map with nonempty convex values and let $u_n \to u$, $u_n \in X$. Let $x_n \to x$, $x_n \in K(u_n)$, $x \in K(u)$. Then*

$$\liminf_{n \to \infty} T_{K(u_n)}(x_n) \supset T_{K(u)}(x).$$

Proof. Let $x \in K(u)$ and

$$S_{K(u)}(x) := \bigcup_{h > 0} \frac{K(u) - x}{h}.$$

Since $K(u)$ is a convex set and $\liminf_{n \to \infty} T_{K(u_n)}(x_n)$ being the Kuratowski lower limit is closed it follows that it is enough to prove the inclusion

$$S_{K(u)}(x) \subset \liminf_{n \to \infty} T_{K(u_n)}(x_n),$$

see [3]. Let $v \in S_{K(u)}(x)$; i.e., there exists $h > 0$ such that

$$x + h \cdot v \in K(u).$$

Since $K(u) \subset \liminf_{n \to \infty} K(u_n)$ being lower semicontinuous it follows that there exists a sequence $y_n \in K(u_n)$ such that $y_n \to x + h \cdot v$. Let $v_n := (y_n - x_n)/h$. For $0 < h_n < h$

$$x_n + h_n \cdot v_n \in K(u_n)$$

and consequently

$$v_n \in T_{K(u_n)}(x_n).$$

Since $v_n \to v$ it follows that

$$v \in \liminf_{n \to \infty} T_{K(u_n)}(x_n).$$

Remark. In the case when $K(\cdot)$ is continuous, then using the dual version of the Attouch theorem [1], it can be proved

$$\liminf_{\substack{K(u_n) \ni x_n \to x \\ n \to \infty}} T_{K(u_n)}(x_n) = T_{K(u)}(x).$$

LEMMA 2. *Let X be a Banach space and let* $K: X \rightsquigarrow \mathbf{R}^n$ *be a lower semi-continuous set-valued map with nonempty convex closed values and* $u \in X$, $c > 0$. *Then for* $\forall \varepsilon > 0$, $\forall x \in K(u)$, $\exists \eta(x) > 0$ *such that for* $\forall \tilde{u} \in X$, $\|u - \tilde{u}\| < \eta(x)$, $\forall \tilde{x} \in K(\tilde{u})$, $\|x - \tilde{x}\| < \eta(x)$ *holds*

$$\delta(T_{K(u)}(x) \cap c \cdot B,\ T_{K(\tilde{u})}(\tilde{x}) \cap c \cdot B) < \varepsilon.$$

Proof. Let us suppose that Lemma 2 does not hold, i.e., $\exists \varepsilon > 0$, $\exists x \in K(u)$, such that for $\forall n \in \mathbf{N}$, $\exists u_n \in X$, $\|u - u_n\| < 1/n$, $\exists x_n \in K(u_n)$, $\|x - x_n\| < 1/n$, $\delta(T_{K(u)}(x) \cap c \cdot B,\ T_{K(u_n)}(x_n) \cap c \cdot B) > \varepsilon$, i.e., for $\forall n \in \mathbf{N}$ there exists

$$z_n \in T_{K(u)}(x) \cap c \cdot B, \qquad z_n \notin B(T_{K(u_n)}(x_n) \cap c \cdot B, \varepsilon).$$

From the sequence z_n, $n = 1, \ldots$, we choose a convergent subsequence

$$z_n \to z \in T_{K(u)}(x)$$

and

$$z \notin \liminf_{n \to \infty} T_{K(u_n)}(x_n),$$

i.e.,

$$T_{K(u)}(x) \not\subset \liminf_{n \to \infty} T_{K(u_n)}(x_n).$$

We got a contradiction with Lemma 1.

Proof of Theorem 1. Since $F(\cdot)$ is bounded we may define

$$c := 1 + \sup_{x \in \mathrm{bd}(K(u_0))} \sup_{f \in F(x)} \|f\| < \infty.$$

Let $x_0 \in \mathrm{bd}(K(u_0))$. Due to the assumptions there exists

$$f_0 \in F(x_0) \cap \mathrm{int}(T_{K(u_0)}(x_0))$$

and $1 > \varepsilon_1(x_0) > 0$ such that

$$B(f_0, \varepsilon_1(x_0)) \subset \mathrm{int}(T_{K(u_0)}(x_0)).$$

Since F is lower semicontinuous in x_0, we get that for $\varepsilon_1(x_0)/4$ there exists $\eta(x_0) > 0$ such that

$$\forall x \in B(x_0, \eta(x_0)), \qquad F(x) \cap B(f_0, \varepsilon_1(x_0)/4) \neq \emptyset.$$

From Lemma 2 it follows that there exists $\varepsilon_2(x_0) > 0$ such that if $\|u - u_0\| < \varepsilon_2(x_0)$ and $\|\tilde{x} - x_0\| < \varepsilon_2(x_0)$, $\tilde{x} \in K(u)$, then

$$\delta(T_{K(u_0)}(x_0) \cap c \cdot B, T_{K(u)}(\tilde{x}) \cap c \cdot B) < \varepsilon_1(x_0)/4.$$

Since $K(u_0)$ is a compact set there exist points $x_i \in \mathrm{bd}(K(u_0))$, $i = 1, ..., N$, such that

$$\mathrm{bd}(K(u_0)) \subset \bigcup_{i=1}^{N} B(x_i, \min(\eta(x_i), \varepsilon_2(x_i))).$$

Let

$$\varepsilon_2 := \min_{i=1, ..., N} \varepsilon_2(x_i).$$

Let

$$M := K(u_0) \cup \bigcup_{i=1}^{N} B(x_i, \min(\eta(x_i), \varepsilon_2(x_i))).$$

Since M is an open set there exists $\eta > 0$ such that

$$B(K(u_0), \eta) \subset M,$$

i.e., $\forall x \in B(\mathrm{bd}(K(u_0)), \eta)$ there exists $i \in \{1, ..., N\}$, such that

$$x \in B(x_i, \min(\eta(x_i), \varepsilon_2(x_i))).$$

Since $K(\cdot)$ is lower semicontinuous it follows that there exists $\varepsilon_3 > 0$ such that if $\|u - u_0\| < \varepsilon_3$ then $K(u) \cap B(K(u_0), \eta) \neq \emptyset$ and consequently

$$\delta(K(u) \cap B(K(u_0), \eta), K(u_0)) < \eta.$$

Moreover, since the map $K(u) \cap B(K(u_0), \eta)$ is lower semicontinuous for $\|u - u_0\| < \varepsilon_3$, see [5], it follows that there exists $\varepsilon_4 > 0$, $\varepsilon_3 > \varepsilon_4$ such that

$$\|u - u_0\| < \varepsilon_4 \Rightarrow \delta(K(u_0), K(u) \cap B(K(u_0), \eta)) < \eta.$$

It follows that

$$\|u - u_0\| < \varepsilon_4 \Rightarrow d^*(K(u_0), K(u) \cap B(K(u_0), \eta)) < \eta.$$

If we assume that $K(\cdot)$ is continuous then again there exists $\varepsilon_4 > 0$ such that

$$\|u - u_0\| < \varepsilon_4 \Rightarrow d^*(K(u), K(u_0)) < \eta.$$

It follows that for every $\tilde{x} \in \mathrm{bd}(K(u) \cap B(K(u_0), \eta))$ ($\tilde{x} \in \mathrm{bd}(K(u))$ in the continuous case, respectively) there exists $x_i \in \mathrm{bd}(K(u_0))$ such that

$$\|\tilde{x} - x_i\| < \min(\eta(x_i), \varepsilon_2(x_i)).$$

Let $\|u - u_0\| \leqslant \varepsilon := \min(\varepsilon_2, \varepsilon_4)$. For every $\tilde{x} \in \mathrm{bd}(K(u) \cap B(K(u_0), \eta))$ ($\tilde{x} \in \mathrm{bd}(K(u))$ in the continuous case, respectively) there exists $x_i \in \mathrm{bd}(K(u_0))$ such that

$$F(\tilde{x}) \cap B(f_i, \varepsilon_1(x_i)/4) \neq \emptyset.$$

Let

$$\tilde{f} \in F(\tilde{x}) \cap B(f_i, \varepsilon_1(x_i)/4).$$

We prove

$$B(\tilde{f}, \varepsilon_1(x_i)/4) \subset T_{K(u)}(\tilde{x}). \tag{10}$$

Let $e \in S$. It follows that

$$\tilde{f} + 3/4 \cdot \varepsilon_1(x_i) \cdot e \in B(\tilde{f}, 3/4 \cdot \varepsilon_1(x_i)) \subset T_{K(u_0)}(x_i) \cap c \cdot B.$$

Since

$$\delta(T_{K(u_0)}(x_i) \cap c \cdot B, T_{K(u)}(\tilde{x}) \cap c \cdot B) < \varepsilon_1(x_i)/4,$$

then there exists $m \in T_{K(u)}(\tilde{x}) \cap c \cdot B$ such that

$$\|\tilde{f} + 3/4 \cdot \varepsilon_1(x_i) \cdot e - m\| < \varepsilon_1(x_i)/4.$$

Therefore

$$\sigma_{T_{K(u)}(\tilde{x})}(e) \geq \langle m, e \rangle > \sigma_{B(\bar{f}, 3/4 \cdot \varepsilon_1(x_i))}(e) - \varepsilon_1(x_i)/4$$
$$= \sigma_{B(\bar{f}, 1/2 \cdot \varepsilon_1(x_i))}(e) \tag{11}$$

and (10) follows. Consequently

$$\bar{f} \in T_{K(u)}(\tilde{x}).$$

It follows that

$$F(\tilde{x}) \cap T_{K(u)}(\tilde{x}) \neq \emptyset.$$

For $\tilde{x} \in \operatorname{int}(K(u) \cap B(K(u_0), \eta))$ ($\tilde{x} \in \operatorname{int}(K(u))$ in the continuous case, respectively) obviously

$$F(\tilde{x}) \cap T_{K(u)}(\tilde{x}) \neq \emptyset.$$

An Application in the Theory of Control Systems with State Constraints. Now we go back to the beginning of the paper to show how Theorem 1 can be used in the framework of the theory of control systems with state constraints.

Let $V: X \rightsquigarrow Z$, $f: \operatorname{Graph}(V) \mapsto X$, where X, Z are finite dimensional spaces and $K \subset \operatorname{Dom}(V) := \{x \in X \mid V(x) \neq \emptyset\}$. Let us regard a control system (1) with state constraints (2). Then the regulation map is defined as

$$R_K(x) := \{v \in V(x) \mid f(x, v) \in T_K(x)\}.$$

Let us assume that $V(\cdot)$ is upper semicontinuous, f is continuous, $F(x) := \{f(x, v) \mid v \in V(x)\}$ has convex values, and $f(\cdot, \cdot)$, $V(\cdot)$ have linear growth. Then the existence of the viable solutions to control system is equivalent with $R_K(x) \neq \emptyset$, $\forall x \in K$, see [2, 3].

If we assume that $V(\cdot)$ is continuous on the boundary of the set K and $\forall x \in K$, $\tilde{R}_K(x) \neq \emptyset$ where $\tilde{R}_K(x) := \{v \in V(x) \mid f(x, v) \in \operatorname{int}(T_K(x))\}$ then from Theorem 1 it follows that the control problem has a viable solution even for a small perturbation of the viability set K.

Acknowledgments

The author is greatly indebted to I. Vrkoč and J.-P. Aubin for numerous comments on the manuscript.

REFERENCES

1. H. ATTOUCH, "Variational Convergence for Functions and Operators," Pitman, London, 1984.
2. J. P. AUBIN, Viability theory, to appear.
3. J. P. AUBIN AND A. CELLINA, "Differential Inclusions," Springer, Berlin, 1984.
4. L. CESARI, "Optimization-Theory and Applications," Springer, Berlin, 1983.
5. E. KLEIN AND A. THOMPSON, "Theory of Correspondences," Wiley–Interscience, New York, 1984.
6. C. KURATOWSKI, "Topologie," Panstwowe Wydawnictwo Naukowe, Warszava, 1958.
7. A. KURZHANSKI, "Control and Estimation under Uncertainty," Nauka, Moscow, 1978. [In Russian]
8. A. KURZHANSKI, Dynamic control system estimation under uncertainty conditions, I, *Problems Control Inform. Theory* **9** (1980), 395–406.
9. A. KURZHANSKI, Dynamic control system estimation under uncertainty conditions, II, *Problems Control Inform. Theory* **10** (1981), 33–42.
10. G. LEITMANN, Deterministic control of uncertain systems, *Astronautica Acta* **7** (1980), 1457–1461.
11. G. LEITMANN, On the efficacy of nonlinear control in uncertain linear systems, *J. Dynamic Systems Meas. Control* **103** (1981), 95–102.
12. G. LEITMANN AND E. P. RYAN, Output feedback control of a class of singularly perturbed uncertain dynamical systems, *in* "Proceedings, American Control Conference, 1987," pp. 1590–1594.
13. G. LEITMANN, E. P. RYAN, AND A. STEINBERG, Feedback control of uncertain systems: Robustness with respect to neglected actuator and sensor dynamics, *Internat. J. Control* **43** (1986), 1243–1256.
14. U. MOSCO, Convergence of convex sets and of solutions to variational inequalities, *Adv. in Math.* **3** (1969), 510–585.