Progress in Systems and Control Theory
Volume 16

Series Editor
Christopher I. Byrnes, Washington University

Set-valued Analysis and Differential Inclusions

A Collection of Papers resulting from a Workshop held in Pamporovo, Bulgaria, September, 1990

Alexander B. Kurzhanski
Vladimir M. Veliov
Editors

IIASA
International Institute for Applied Systems Analysis
Laxenburg, Austria

Birkhäuser
Boston · Basel · Berlin

[20] VINTER R.B. WOLENSKI P. (1990): *Hamilton-Jacobi theory for optimal control problems with data measurable in time*, *SIAM J. Control and Optimization*, 28 No 6, 1404-1420.

Hélène Frankowska
CEREMADE, Université Paris-Dauphine
75775 Paris, FRANCE

Sławomir Plaskacz
Instytut Matematyki, Uniwersytet Mikołaja Kopernika
ul. Chopina 12/18, PL 87-100 Toruń, POLAND

Tadeusz Rzeżuchowski
Instytut Matematyki, Politechnika Warszawska
pl. Politechniki 1, PL 00-661 Warszawa, POLAND

Projection of Differential Equations

Vlastimil Křivan*

Abstract

In the paper a class of projections, called the G-projections, is defined. These projections are used to project the dynamics of a differential inclusion $\dot{x}(t) \in F(x(t))$ onto the contingent cone to a given set K. The existence of a solution to the projected differential inclusion is proven. The G-projections generalize the projection of the best approximation, and the G-projected differential inclusions were used to construct models in population biology.

1 Introduction

Projections of differential inclusions play an important role in many applications of mathematics. For example, projected differential inclusions were used in mechanics, see [11], or in economics to build planning models, see [5],[6],[7],[8]. In these applications the projection of the best approximation was used to project the dynamics of a differential inclusion or an equation onto the tangent cone to a given set. But in some cases the projection of the best approximation may not be adequate. For this reason, we define in this paper a class of projections, called the G-projections that generalize the projection of the best approximation. The G-projection of differential inclusions was motivated by the construction of population growth equations, see [9],[10]. Using the G-projection, the dynamics of a differential inclusion

(1) $$\dot{x}(t) \in F(x(t))$$

is projected onto the contingent cone $T_K(x)$ to a given set K. We get a projected differential inclusion

(2) $$\dot{x}(t) \in \Pi^G_{T_K}(F(x(t)))$$

*Partially supported by CAS 18002

where $\Pi^G_{T_K}(F(x))$ denotes the G-projection of $F(x)$ onto the contingent cone $T_K(x)$. Unfortunately, the standard existence theorems for differential inclusions cannot be used to prove the existence theorem for (2), since in general, $\Pi^G_{T_K}(F(\cdot))$ does not inherit the properties of $F(\cdot)$ and K. To prove an existence theorem for (2), an existence theorem for a generalized differential variational inequality is given and it is proven that the projected differential inclusion (2) has the same solution set as this variational inequality. A similar approach to prove an existence theorem for projected differential inclusions in the case of the projection of best approximation was used in [1].

2 The G-projection

Definition 1. Let $A \subset \mathbf{R}^n$. Then $C_+(A)$ denotes the positive cone spanned by A, i. e.,

$$C_+(A) := \begin{cases} \bigcup_{k\geq 0} kA & \text{if } A \neq \emptyset \\ \{0\} & \text{if } A = \emptyset. \end{cases}$$

Remark. Let $g \in \mathbf{R}^n$. Instead of writing $C_+(\{g\})$ we will write $C_+(g)$.

Let us recall that for a non-empty set $A \subset \mathbf{R}^n$ the negative polar cone is $A^- := \{y \in \mathbf{R}^n \mid \langle y, a\rangle \leq 0, \text{ for every } a \in A\}$. For a set-valued map $F : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ we denote by $\mathrm{Im}(F)$ its image, by $\mathrm{Dom}(F)$ its domain and by $\mathrm{Graph}(F)$ its graph.

Lemma 2.1 *Let $G : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ be a set-valued map with convex compact values.*

a) *Let $x_o \in \mathrm{Dom}(G)$ and $0 \notin G(x_o)$. Then $C_+(G(x_o))$ is the smallest closed convex cone containing the set $G(x_o)$. Consequently $C_+(G(x_o)) = (G(x_o))^{--}$.*

b) *Let $\mathrm{Graph}(G)$ be compact and $0 \notin \mathrm{Im}(G)$. Then the set-valued map $C_+(G) : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ has a closed graph.*

Proof. a) Let $x_o \in \mathrm{Dom}(G)$. Since $C_+(G(x_o))$ is a cone spanned by a convex compact set disjoint from 0, it follows that it is closed. Therefore $C_+(G(x_o)) = (G(x_o))^{--}$, see [1,p.31].

b) We prove that $C_+(G(.))$ has closed graph by contradiction. Let $(x_n, y_n) \in \mathrm{Graph}(C_+(G))$, $(x_n, y_n) \longrightarrow (x, y) \notin \mathrm{Graph}(C_+(G))$. Since $y \notin C_+(G(x))$, i. e. $y \neq 0$, it follows that $y_n = 0$ only for a finite number of n's. Therefore we may assume that $x_n \in \mathrm{Dom}(G)$ for every n. It follows there exist $k_n \geq 0$ and $g_n \in G(x_n)$ such that $y_n = k_n g_n$. Since $G(.)$ has compact graph, we can choose a convergent subsequence from g_n (denoted again g_n) such that $g_n \longrightarrow g \in G(x)$. Since $0 \notin \mathrm{Im}(G)$ it follows that a subsequence of k_n is converging, $k_n \longrightarrow k < \infty$. Consequently, $y = kg$. Hence $y \in C_+(G(x))$, a contradiction.

Q.E.D.

In the following definition a class of projections, called the G-projections is defined. These projections "project" a point onto a set along the directions given by a set G.

Definition 2. Let $K \subset \mathbf{R}^n$ be a non-empty set, $G \subset \mathbf{R}^n$ be possibly empty. Then

1. For every $g \in G$ and every $u \in C_+(g) + K$ define
$$k^K_g(u) := \inf\{k \geq 0 \mid u - kg \in K\},$$
$$\Pi^g_K(u) := u - k^K_g(u)g.$$
2. Let $u \in C_+(G) + K$. Then
$$\Pi^G_K(u) := \bigcup_{\{g\in G \mid u \in C_+(g)+K\}} \Pi^g_K(u).$$
3. If $G = \emptyset$, then we set
$$\Pi^G_K(u) := u.$$

We say that $\Pi^G_K(u)$ is the G-projection of u onto the set K.

In the rest of this paper the set K from Definition 2 will be the contingent cone, see [1].

3 Generalized Variational Differential Inequalities

Let $K \subset \mathbf{R}^n$ be a non-empty closed set. Let us consider a set-valued map $F : K \rightsquigarrow \mathbf{R}^n$ and let

$$\forall x \in K,\ F(x) \cap T_K(x) \neq \emptyset. \tag{3}$$

For upper semicontinuous, convex and compact valued map $F(\cdot)$, (3) is well known viability condition ensuring the existence of a viable solution (in K) for the following differential inclusion

$$\dot{x}(t) \in F(x(t)), \tag{4}$$

see [1],[2].

If (3) is not satisfied then the dynamics of (4) has to be changed at least on the set of these points where (3) does not hold in order to get a viable solution in K. Let $G : K \rightsquigarrow \mathbf{R}^n$ be given set-valued map and let us consider the following viability problem

$$\left.\begin{array}{rcll} \dot{x}(t) & \in & F(x) - C_+(G(x(t))) & \text{for almost all } t \in [0,T] \\ x(t) & \in & K & \text{for every } t \in [0,T] \\ x(0) & = & x_0 \in K. & \end{array}\right\} \tag{5}$$

Let

$$\Omega(x) := F(x) \cap (T_K(x) + C_+(G(x))), \quad x \in K. \tag{6}$$

Next theorem gives an existence result for (5).

Theorem 3.1 *Let $K \subset \mathbf{R}^n$ be a non-empty compact set, $F : K \rightsquigarrow \mathbf{R}^n$ be an upper semicontinuous set-valued map with non-empty compact convex values. Let $G : K \rightsquigarrow \mathbf{R}^n$ be a set-valued map with compact graph, convex values and $0 \notin \mathrm{Im}(G)$. Let $\mathrm{Dom}(\Omega) = K$ and*

$$\sup_{x \in K} \inf_{f \in \Omega(x)} \inf_{z \in \Pi^{G(x)}_{T_K(x)}(f)} \|f - z\| < c < \infty. \tag{7}$$

Then for every $T > 0$ there exists a solution to (5).

Proof. Let
$$M(x) := F(x) - (\bar{B}(0,c) \cap C_+(G(x))),$$
where $\bar{B}(0,c)$ denotes the closed ball with radius c, centered at 0. Obviously, $M(\cdot)$ has convex and compact values. The set-valued map

$$x \rightsquigarrow \overline{B}(0,c) \cap C_+(G(x))$$

is upper semicontinuous due to Lemma 2.1. Since $F(\cdot)$ is upper semicontinuous and K is compact, it follows (see [1, p.42]) that $\mathrm{Im}(F)$ and consequently $\mathrm{Im}(M)$ are compact sets. It is easy to see that $M(\cdot)$ has closed graph. Indeed, let $m_n \in M(x_n)$, $x_n \longrightarrow x, m_n \longrightarrow m$, i.e., $m_n = f_n - z_n$

where $f_n \in F(x_n)$, $z_n \in \bar{B}(0,c) \cap C_+(G(x_n))$. From compactness and upper semicontinuity it follows that we may choose subsequences converging to $f \in F(x)$ and $z \in \overline{B}(0,c) \cap C_+(G(x))$. Consequently $m = f - z \in M(x)$. Therefore $M(\cdot)$ is upper semicontinuous, having closed graph and compact image. Since

$$\forall x \in K, \quad \exists f \in \Omega(x), \ \exists z \in \Pi^{G(x)}_{T_K(x)}(f) \text{ such that } \|f - z\| < c$$

then we have

$$z \in T_K(x) \cap \{f - (\overline{B}(0,c) \cap C_+(G(x)))\} \subset T_K(x) \cap M(x),$$

so that the tangential condition

$$M(x) \cap T_K(x) \neq \emptyset \tag{8}$$

is satisfied for every $x \in K$. The existence of a viable solution to

$$\dot{x}(t) \in M(x(t))$$

follows from the viability existence theorem, see [1], p.182.

Q.E.D.

Remark. Differential variational inequality, see [1]

$$\left.\begin{array}{rcl} \dot{x}(t) & \in & F(x(t)) - N_K(x(t)), \\ x(t) & \in & K \end{array}\right\} \tag{9}$$

where $N_K(x)$ denotes the normal cone, can be thought as a special case of (5) for $C_+(G(x)) := N_K(x)$ for every $x \in K$.

4 Projected Differential Inclusions

Let $K \subset \mathbf{R}^n$ be a non-empty set and let $F : K \rightsquigarrow \mathbf{R}^n$ be a set-valued map with non-empty values. Let $G : K \rightsquigarrow \mathbf{R}^n$ be a set-valued map and $\mathrm{Dom}(\Omega) = K$, where $\Omega(\cdot)$ is defined by (6). The G-projection of a differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{10}$$

is defined in the following way.

Definition 3. The G-projection of the differential inclusion (10) is the following differential inclusion

$$\dot{x}(t) \in \Pi^G_{T_K}(F(x(t))) := \bigcup_{f \in \Omega(x(t))} \Pi^{G(x(t))}_{T_K(x(t))}(f). \tag{11}$$

This differential inclusion is called the *projected differential inclusion.*

The following theorem shows that solutions to the differential inclusion (11) are solutions to the differential inclusion (5) and conversely.

Let us recall that a set $K \subset \mathbf{R}^n$ is *regular* if Bouligand contingent cone and Clarke tangent cone coincide and, consequently, they are convex cones, see [4]. If $x \rightsquigarrow T_K(x)$ is lower semicontinuous then the set K is regular, see [3]. In [3] such sets were called *sleek* sets.

Theorem 4.1 *Let $K \subset \mathbf{R}^n$ be a non-empty regular set. Let $F : K \rightsquigarrow \mathbf{R}^n$ be a set-valued map with non-empty values, $G : K \rightsquigarrow \mathbf{R}^n$. Let $\mathrm{Dom}(\Omega) = K$ and for every $x \in K$,*

$$G(x) \cap T_K(x) = \emptyset. \tag{12}$$

Then the viable solutions to the differential inclusion (5) are the solutions to the differential inclusion (11) and conversely.

Proof. Since

$$\Pi^G_{T_K}(F(x(t))) \subseteq F(x(t)) - C_+(G(x(t)))$$

then solutions to the differential inclusion (11) are solutions to the differential inclusion (5).

Conversely: every viable solution to the differential inclusion (5) is a solution to the differential inclusion (11). Let $x(\cdot)$ be a viable solution to the differential inclusion (5) on $[0,T]$, $(T > 0)$ i.e.

$$\dot{x}(t) = f(t) - z(t) \quad \text{for a.a. } t \in [0,T],$$

where $f(t) \in F(x(t)), z(t) \in C_+(G(x(t)))$. Let $[0,T] = E_1 \cup E_2$ where $t \in E_1$ if $G(x(t)) = \emptyset$ and $t \in E_2$ if $G(x(t)) \neq \emptyset$. Let us assume that $x(.)$ is not a solution to the differential inclusion (11). It means that there exists a set $\Lambda \subseteq [0,T]$ of a positive Lebesque measure $\mu(\Lambda) > 0$ such that

$$\dot{x}(t) \notin \Pi^G_{T_K}(F(x(t))) \quad \text{for} \quad t \in \Lambda$$

Since for $t \in E_1$, $\dot{x}(t) = f(t) \in \Pi^G_{T_K}(F(x(t)))$ it follows

$$\mu(\Lambda \cap E_1) = 0,$$
$$\mu(\Lambda \cap E_2) = \mu(\Lambda) > 0.$$

For almost all $t \in \Lambda \cap E_2$,

$$\dot{x}(t) = f(t) - k(t)g(t) \in T_K(x(t))$$

where $g(t) \in G(x(t))$ and $k(t) > k^{T_K(x(t))}_{g(t)}(f(t))$. For almost all $t \in \Lambda \cap E_2$

$$-\dot{x}(t) = k(t)g(t) - f(t) \in T_K(x(t)).$$

Since $f(t) - k^{T_K(x(t))}_{g(t)}(f(t))g(t) \in T_K(x(t))$ for $t \in \Lambda \cap E_2$ and $T_K(x(t))$ is convex (since K is regular), then

$$(k(t) - k^{T_K(x(t))}_{g(t)}(f(t)))g(t) \in T_K(x(t)) \quad \text{for a.e.} \quad t \in \Lambda \cap E_2.$$

Due to the assumption (12), the inequality $k(t) - k^{T_K(x(t))}_{g(t)}(f(t)) > 0$ cannot hold, i. e., $k(t) = k^{T_K(x(t))}_{g(t)}(f(t))$ for almost all $t \in \Lambda \cap E_2$ and $x(\cdot)$ is a solution to the differential inclusion (11).

Q.E.D.

5 Projected Differential Inclusions on the Sets Defined by Constraints

In this section it is assumed that the set K is defined by p functions $r_i(\cdot)$, $i = 1, \ldots, p$,

$$K := \{x \in \mathbf{R}^n \mid r_1(x) \leq 0, \ldots, r_p(x) \leq 0\}. \tag{13}$$

We consider again differential inclusion (10). For K defined by (13) we may define a set-valued selection from $\Pi^G_{T_K}(F(x))$, denoted by $\pi(F(x))$ such that the solutions to

$$\dot{x}(t) \in \pi(F(x(t)))$$

are still solutions to (5) and conversely.

Throughout this section it is assumed that $r_i(\cdot)$, $i = 1, \ldots, p$ are strictly differentiable, see [4]. This is for example satisfied if $r_i(\cdot)$, $i = 1, \ldots, p$

are continuously differentiable. By $r_i'(x)$ we denote the strict derivative of $r_i(x)$. Let

$$I(x) := \{i = 1, \ldots, p \mid r_i(x) = 0\}.$$

If we assume that $r_i'(x)$, $i \in I(x)$ are positively linearly independent then it follows from [4] that

$$T_K(x) = \{v \in \mathbf{R}^n \mid \langle r_i'(x), v\rangle \leq 0, \ \ i \in I(x)\}.$$

Let $G_i : K \rightsquigarrow \mathbf{R}^n$, $i = 1, \ldots, p$ be given set-valued maps. For every $x \in K$ we define:

$$G(x) := \mathrm{conv}\{G_i(x) \mid i \in I(x) \ \text{ such that } x \in \mathrm{Dom}(G_i)\}. \tag{14}$$

Let

$$\omega(x) := \{f \in F(x) \mid \exists z \in C_+(G(x)), \langle r_i'(x), f - z\rangle = 0, i \in I(x)\}. \tag{15}$$

Now we define a set-valued map $\pi(F(.))$.

Definition 4. Let $K \subset \mathbf{R}^n$ defined by (13) be a non-empty set where the functions $r_i(.)$, $i = 1, \ldots, p$ are strictly differentiable. Let for every $x \in K$, $r_i'(x)$, $i \in I(x)$ be positively linearly independent and $F : K \rightsquigarrow \mathbf{R}^n$, $G_i : K \rightsquigarrow \mathbf{R}^n$, $i = 1, \ldots, p$ be set-valued maps. Then for all $x \in K$, we define:

i) Let $f \notin T_K(x)$ and $f \in \omega(x)$ then

$$\pi(f) := \{f - z \mid z \in C_+(G(x)), \langle r_i'(x), f - z\rangle = 0, i \in I(x)\}$$

ii) Let $f \notin T_K(x)$, $f \notin \omega(x)$ and $f \in \Omega(x)$. Let p be any element of $\Pi^G_{T_K}(f)$. Then

$$\pi(f) := p.$$

iii) Let $f \in T_K(x)$ then

$$\pi(f) := f.$$

iv)

$$\pi(F(x)) := \{\pi(f) \mid f \in \Omega(x)\}.$$

Theorem 4.1 may be reformulated for the differential inclusion

$$\dot{x}(t) \in \pi(F(x(t))). \tag{16}$$

Theorem 5.1 *Let $K \subset \mathbf{R}^n$ defined by (13) be a non-empty set. Let $F : K \rightsquigarrow \mathbf{R}^n$ be a set-valued map with non-empty values, $G_i : K \rightsquigarrow \mathbf{R}^n$, $i = 1, \ldots, p$. Let $G(x) \cap T_K(x) = \emptyset$ for every $x \in K$ and $Dom(\Omega) = K$. Then the solutions to the differential inclusion (16) are the viable solutions to the differential inclusion (5) and conversely. Moreover, if $F(x) = \{f(x)\}$ is single valued and for every $x \in K$ and every $f \in F(x) \setminus T_K(x)$,*

$$(C_+(G(x)) - C_+(G(x))) \cap T_K(x) \cap (-T_K(x)) = \{0\} \tag{17}$$

then $\pi(f(x))$ is single-valued.

Proof. First we prove that for all $x \in K$

$$\pi(F(x)) \subseteq \Pi^G_{T_K}(F(x)). \tag{18}$$

Let us assume that there exist $f \in \Omega(x)$, $k > 0$, $g \in G(x)$ such that $f - kg \in \pi(f)$ and $f - kg \notin \Pi^G_{T_K}(F(x))$. Consequently, $f \notin T_K(x)$ and $f \in \omega(x)$, otherwise (18) is obviously satisfied. It follows $k > k_g^{T_K}$. Since $\langle f - kg, r_i'(x)\rangle = 0$ for $i \in I(x)$ and $\langle f - k_g^{T_K} g, r_i'(x)\rangle \leq 0$ for every $i \in I(x)$ we get

$$\langle (k - k_g^{T_K})g, r_i'(x)\rangle \leq 0, \ \ i \in I(x).$$

Therefore $(k - k_g^{T_K})g \in T_K(x)$. This contradicts with the assumption $G(x) \cap T_K(x) = \emptyset$. Therefore $k = k_g^{T_K}$ and $\pi(F(x)) \subseteq \Pi^G_{T_K}(F(x)) \subset F(x) - C_+(G(x))$. It follows that the solutions to the differential equation (16) are the solutions to the differential inclusion (5).

Following the lines of the proof of Theorem 4.1, we assume that for a solution $x(.)$ to (5) there exists a set Λ of a positive Lebesque measure such that

$$\dot{x}(t) \notin \pi(F(x(t))), \ \ t \in \Lambda.$$

Let E_1 and E_2 be as in the proof of Theorem 4.1. Since for $t \in E_1$, $\dot{x}(t) = f(t)$ and for almost all $t \in E_1$, $f(t) \in T_K(x(t))$, it follows $f(t) \in \pi(f(t))$ for almost all $t \in E_1$. Consequently, $\mu(\Lambda \cap E_1) = 0$ and $\mu(\Lambda \cap E_2) > 0$. Since $x(t)$ is viable, for almost all $t \in \Lambda \cap E_2$

$$\dot{x}(t) = f(t) - k(t)g(t) \in T_K(x(t)) \tag{19}$$

where $f(t) \in F(x(t))$, $g(t) \in G(x(t))$ and $k(t) > 0$. For almost all $t \in \Lambda \cap E_2$

$$-\dot{x}(t) = k(t)g(t) - f(t) \in T_K(x(t)). \tag{20}$$

From (19), (20) it follows that for almost all $t \in \Lambda \cap E_2$,

$$\langle r_i'(x(t)), f(t) - k(t)g(t)\rangle = 0, \ \ \text{for every} \ \ i \in I(x(t)).$$

It follows that for almost all $t \in \Lambda \cap E_2$, $f(t) \in \omega(x(t))$ and $f(t) - k(t)g(t) \in \pi(f(t))$. Consequently, $x(t)$ is solution to (16).

We prove that for the single valued map $F(x) = \{f(x)\}$, $\pi(f(x))$ is single-valued too. Since for $f \in T_K(x)$ the statement is trivial, let us assume that $f \notin T_K(x)$. We may assume that $f \in \omega(x)$; otherwise $\pi(f)$ is single-valued. Let

$$z^1, z^2 \in \pi(f), \quad z^1 \neq z^2.$$

From Definition 4 it follows

$$z^1 = f - k^1 g^1, \quad z^2 = f - k^2 g^2,$$

where

$$k^1, k^2 > 0, \quad g^1, g^2 \in G(x),$$

and

$$\langle r_i'(x), f - k^1 g^1 \rangle = \langle r_i'(x), f - k^2 g^2 \rangle = 0, \quad i \in I(x).$$

Therefore,

$$\langle r_i'(x), k^1 g^1 - k^2 g^2 \rangle = 0, \quad i \in I(x)$$

and consequently

$$k^1 g^1 - k^2 g^2 \in T_K(x) \cap -T_K(x).$$

From the assumptions it follows

$$k^1 g^1 = k^2 g^2,$$

i.e., $\pi(f)$ is single valued.

Q.E.D.

Let us note that (16) generalizes the following differential inclusion

$$\dot{x}(t) \in \pi^{\perp}(F(x(t))) \tag{21}$$

where $\pi^{\perp}$ denotes the projection of best approximation onto $T_K(x)$.

Proposition 5.1 *Let $K \subset \mathbf{R}^n$ defined by (13) be a non-empty set where the functions $r_i(.)$, $i = 1, \ldots, p$ are strictly differentiable and $r_i'(x)$, $i \in I(x)$ are positively linearly independent for every $x \in K$. Let $G_i(x) := \{r_i'(x)\}$ if $r_i(x) = 0$. Then (16) has the same solution set as (21).*

Proof. Let us note that under the assumptions it follows that the normal cone to K is

$$N_K(x) = \{ \sum_{i \in I(x)} \alpha_i r_i'(x) \mid \alpha_i \geq 0\}$$

and $T_K(x)$ is convex, see [4], p.57. From Theorem 5.1 it follows that (16) has the same solution set as (5) which is

$$\dot{x}(t) \in F(x(t)) - N_K(x(t)).$$

From [1] it follows that this differential inclusion has the same solution set as (21).

Q.E.D.

6 $G-$projection of control systems

The method of $G-$projection can also be used to "correct" the dynamics of control systems when there is no control regulating a viable solution. Let us consider a control system

$$\begin{array}{rcl} x'(t) &=& f(x(t), u(t)) \\ u(t) &\in& U \\ x(t) &\in& K, \end{array} \tag{22}$$

where $U \subset \mathbf{R}^l$, $K \subset \mathbf{R}^n$, $f : K \times U \mapsto \mathbf{R}^n$.

From the Filippov Lemma, [1,p.91] it follows that for continuous function $f(\cdot, \cdot)$, the control system (22) is equivalent to the following constrained differential inclusion

$$\begin{array}{rcl} \dot{x}(t) &\in& F(x(t)) := \{f(x(t), u(t)) \mid u(t) \in U\} \\ x(t) &\in& K. \end{array}$$

Consequently, the results from the previous sections may be reformulated for the control system (22).

Let us remark that using the contingent derivative instead of the contingent cone (see [1]) all the results stated for the autonomous case here may be reformulated to the non-autonomous case.

REFERENCES

[1] J.P. Aubin and A. Cellina (1984) *Differential Inclusions.* Springer, Berlin.

[2] J.P. Aubin (1991) *Viability Theory.* Birkäuser.

[3] J.P. Aubin and H. Frankowska (1990) *Set-valued Analysis.* Birkäuser, Boston.

[4] F.H. Clarke (1983) *Optimization and Nonsmooth Analysis.* Wiley Interscience, New York.

[5] B. Cornet (1983) Existence of slow solutions for a class of differential inclusions. *J. Math. Anal. Appl.* **96**, pp. 130-147.

[6] B. Cornet (1981) *Contributins a la theorie mathematique des mecanismes dynamiques d'allocation des resources.* These de doctoral d'etat. Universite de Paris-Dauphine.

[7] C. Henry (1972) Differential Equations with Discontinuous Right Hand Side for Planning Procedures. *J. Econ. Theory,* **4**, pp. 554-551.

[8] C. Henry (1973) An Existence Theorem for a Class of Differential Equations with Multivalued Right Hand Side. *J. Math. Anal. Appl.* **41**, pp. 179-186.

[9] A. Kastner-Maresch and V. Křivan. Discontinuous Differential Equations in Popular Biology (to appear).

[10] V. Křivan (1991) Construction of population growth equations in the presence of viability constraints. *J. Mathematical Biology* **29**, pp. 379-387.

[11] J.-J. Moreau (1977) Evolution problem associated with a moving convex set in Hilbert space. *J. Diff. Equations* **26**, pp. 347-374.

Vlastimil Křivan
South Bohemian Biological Research Center
Czech and Slovak Academy of Sciences
370 05 České Budějovice
Czech and Slovak Federal Republic

Modified Euler Methods for Differential Inclusions

Frank Lempio

Abstract

Classical Euler method and simple modifications like the method of Euler-Cauchy, improved Euler method and implicit midpoint rule are discussed with regard to the approximate solution of differential inclusions.

Numerical tests suggest first order convergence of Euler's method at least for specially structured right-hand sides even if the usual Lipschitz condition does not hold. The basic idea of the proof of this convergence property is sketched using a strengthened one-sided Lipschitz condition.

Order reduction for methods which are of higher order for single-valued sufficiently smooth right-hand sides is exemplified numerically for improved Euler method and implicit midpoint rule. Typical advantages of implicit midpoint rule are discussed.

(AMS) Subject Classification: 34A60, 49D25, 65L05
Keywords: differential inclusions, difference methods.

1 Introduction and Preliminaries

Our aim is to study convergence properties of difference methods for differential inclusions. In this paper, we concentrate on simple modifications of Euler's method for the following class of initial value problems.

1.1 Initial Value Problem. Let $I = [t_0, T]$ with $T > t_0$ be a real interval, let $F : I \times \mathbb{R}^n \Rightarrow \mathbb{R}^n$ be a set-valued mapping of $I \times \mathbb{R}^n$ into the set of all subsets of $\mathbb{R}^n$, and let the initial vector $y_0 \in \mathbb{R}^n$ be fixed.

Find an absolutely continuous function $y(\cdot) : I \rightarrow \mathbb{R}^n$ which satisfies the initial condition

$$y(t_0) = y_0 \tag{1.1}$$