65. A commercial gardener wants to feed plants a very specific mix of nitrates and phosphates. Two kinds of fertilizer, Brand *A* and Brand *B*, are available, each sold in 50 pound bags, with the following quantities of each mineral per bag:

	Phosphate	*Nitrate*
Brand *A*	2.5 lbs	10 lbs
Brand *B*	5.0	5

The gardener wants to put at least 30 lbs of nitrates and 15 lbs of phosphates on the gardens and not more than 250 lbs of fertilizer altogether. If Brand *A* costs $8.50 a bag and Brand *B* costs $3.50 a bag, how many bags of each would minimize fertilizer costs?

66. Repeat Exercise 65 if the cost of Brand *B* fertilizer increases to $6.00 a bag.

9.5   DETERMINANTS

. . . A staggering paradox hits us in the teeth. For abstract mathematics happens to work. It is the tool that physicists employ in working with the nuts and bolts of the universe! There are many examples from the history of science of a branch of pure mathematics which, decades after its invention, suddenly finds a use in physics.

F. David Peat

From childhood on, Shannon was fascinated by both the particulars of hardware and the generalities of mathematics. (He) tinkered with erector sets and radios given him by his father . . . and solved mathematical puzzles supplied by his older sister, Catherine, who became a professor of mathematics.

Claude Shannon

In Section 9.2 we introduced matrices as convenient tools for keeping track of coefficients and handling the arithmetic required to solve systems of linear equations. Matrices are being used today in more and more applications. A matrix presents a great deal of information in compact, readable form. Finding optimal solutions to large linear programming problems requires extensive use of matrices. The properties and applications of matrices are studied in *linear algebra,* a discipline that includes much of the material of this chapter. In this section we introduce the determinant of a square matrix as another tool to help solve systems of linear equations.

Dimension (Size) of a Matrix and Matrix Notation

A matrix is a rectangular array arranged in horizontal **rows** and vertical **columns.** The number of rows and columns give the dimension, or size, of the matrix. A matrix with *m* rows and *n* columns is called an ***m* by *n* (*m* × *n*) matrix.** Double subscripts provide a convenient system of notation for labeling or locating matrix entries.

Here are some matrices of various sizes:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{33} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix *A* is 3 × 3, *B* is 3 × 1, and *C* is 2 × 2. *A* and *B* show the use of double subscripts: a_{ij} is the entry in the *i*th row and the *j*th column. The first subscript identifies the row, the second tells the column; virtually all references to matrices are given in the same order, row first and then column. A matrix with the same number of rows and columns is a **square matrix.**

Determinants

Every square matrix A has an associated number called its **determinant,** denoted by $\det(A)$ or $|A|$. To evaluate determinants, we begin by giving a recursive definition, starting with the determinant of a 2×2 matrix, the definition we gave informally in Section 9.1.

Determinant of a 2 $\times$ 2 matrix.   For 2×2 matrix A, we obtain $|A|$ by multiplying the entries along each diagonal and subtracting.

Definition: determinant of a 2 × 2 matrix

For the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the **determinant of A** is given by

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

As in Section 9.1, the easiest way to remember the formula is by visualizing products taken in the direction of two arrows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Thus, for example,

$$\begin{vmatrix} 3 & 2 \\ -4 & 1 \end{vmatrix} = (3)(1) - (2)(-4) = 3 + 8 = 11, \quad \text{and}$$

$$\begin{vmatrix} 9 & 0 \\ 2 & -5 \end{vmatrix} = (9)(-5) - (0)(2) = -45 - 0 = -45.$$

For larger square matrices, the determinant definition uses determinants of smaller matrices within the given matrix. The determinant of a 3×3 matrix uses 2×2 determinants, the determinant of a 4×4 matrix uses 3×3 determinants, and so on.

Minors and cofactors.   We associate with each entry a_{ij} of square matrix A a **minor determinant** M_{ij} and a **cofactor** C_{ij}. The minor determinant, more commonly called simply the **minor,** of an entry is the determinant obtained by deleting the row and column of the entry, so M_{ij} is the determinant we get by crossing out the ith row and the jth column. The cofactor C_{ij} is the signed minor given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

In Example 1, to make it easier to visualize the minor determinant for a given element, we shade the row and column containing that element. When you practice evaluating 3×3 (or larger) determinants, it may help to have a mental picture of a similar shading.

$$\begin{bmatrix} ① & -3 & -2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

(a) Minor M_{11} of a_{11} (unshaded).

$$\begin{bmatrix} 1 & ⊘\!-3 & -2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

(b) Minor M_{12} of a_{12} (unshaded).

$$\begin{bmatrix} 1 & -3 & ⊘\!-2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

(c) Minor M_{13} of a_{13} (unshaded).

▶**EXAMPLE 1 *Finding cofactors*** Find the cofactor for each element in the first row of the matrix.

$$A = \begin{bmatrix} 1 & -3 & -2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

Solution
Follow the strategy. In the first row, $a_{11} = 1$, $a_{12} = -3$, and $a_{13} = -2$. For the minor M_{11}, we delete the shaded row and column in the first margin matrix, leaving the (unshaded) minor $\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix}$ and then use $C_{11} = (-1)^{1+1} M_{11}$.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} = (-1)^2 [0 - (-5)] = 5.$$

To obtain M_{12}, delete row 1 and column 2 (see the second margin in matrix) and then use $C_{12} = (-1)^{1+2} M_{12}$.

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & -1 \\ -1 & 0 \end{vmatrix} = -[0 - (1)] = 1$$

In a similar manner (third margin matrix) C_{13} is given by

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 2 \\ -1 & 5 \end{vmatrix} = [15 - (-2)] = 17 \quad ◀$$

Determinant of a 3 × 3 matrix. The determinant of a 3×3 matrix can be obtained using the elements of the first row.

Definition: cofactor expansion by the first row

Let A be a 3×3 matrix with entries a_{ij}. If C_{ij} and M_{ij} are the cofactor and minor, respectively, of a_{ij} as defined above, then the determinant of A is given by

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}. \quad \textbf{(1)}$$

It is helpful to remember that the cofactors have signs, so that each term of the cofactor expansion of a determinant is a product of three factors: an *entry* a_{ij}, a *sign factor* $(-1)^{i+j}$, and a *minor* M_{ij}. Because the sign factor is either 1 or -1 and depends only on the address (location) of a_{ij}, many people like to use a "sign matrix," that gives the pattern of signs. The sign matrix in the margin may be extended as needed, following the same pattern. Then the above expansion of the determinant has the form

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3 × 3 Sign Matrix

$$|A| = a_{11}(+1)\, M_{11} + a_{12}(-1)\, M_{12} + a_{13}(+1)\, M_{13}.$$
$$\quad\quad\quad \underset{\text{entry}}{\nearrow} \ \underset{\text{sign}}{\uparrow} \ \underset{\text{minor}}{\nwarrow}$$

Determinants of any size have a remarkable property. We get the same number using the entries and cofactors of *any* row or column. For example, each of the following gives the same value for $|A|$ as Equation (1).

$$\text{Expansion by second row} \quad |A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$\text{Expansion by third column} \quad |A| = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$$

To illustrate that the cofactor expansion is independent of the row or column chosen, we return to the matrix from Example 1, for which we already have some cofactors.

▶**EXAMPLE 2** *Cofactor expansion* Evaluate the determinant of matrix A by **(a)** the first row **(b)** the second column.

$$A = \begin{bmatrix} 1 & -3 & -2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

Strategy: **(a)** Since matrix A is the same as the matrix in Example 1, we already have the cofactors for expansion by the first row. Multiply each cofactor by its entry, and add.

Solution

Follow the strategy.

(a) Using $C_{11} = 5$, $C_{12} = 1$, and $C_{13} = 17$ from Example 1, then by Equation (1),

$$|A| = 1 \cdot 5 + (-3) \cdot 1 + (-2) \cdot 17 = 5 - 3 - 34 = -32.$$

(b) Expansion by the second column gives

$$\begin{aligned} |A| &= a_{12} C_{12} + a_{22} C_{22} + a_{32} C_{32} \\ &= (-3)(-1) M_{12} + (2)(+1) M_{22} + (5)(-1) M_{32} \\ &= 3 \begin{vmatrix} 3 & -1 \\ -1 & 0 \end{vmatrix} + 2 \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} - 5 \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ &= 3(-1) + 2(-2) - 5 \cdot 5 = -32, \end{aligned}$$

the same value as for the first-row expansion. ◄

Determinant of an n × n matrix. Since we know how to evaluate 3×3 determinants, we can use a similar cofactor expansion for a 4×4 determinant. Choose any row or column and take the sum of the products of each entry with the corresponding cofactor. The determinant of a 4×4 matrix involves four 3×3 determinants, one for each of the four entries in the chosen row or column. Similarly, the determinant of a 5×5 matrix uses five 4×4 determinants. We give no formal definition of the procedure to evaluate the determinant of an $n \times n$ matrix, but it should be clear from the form of Equation (1). It should also be clear that the number of arithmetic operations required to evaluate a determinant grows staggeringly large as the size of the matrix increases.

Elementary row (column) operations and determinants. One way to simplify the evaluation of determinants is to recognize that certain elementary matrix operations leave the determinant unchanged.

> **Elementary operation property**
>
> Given a square matrix A, if the entries of one row (column) are multiplied by a constant and added to the corresponding entries of another row (column), then the determinant of the resulting matrix is still equal to $|A|$.

Applying the Elementary Operation Property (EOP) may give some zero entries that make the evaluation of a determinant much easier, as illustrated in the next example.

▶**EXAMPLE 3** *Elementary operations*   Evaluate the determinant of the matrix

$$A = \begin{bmatrix} -2 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ -1 & 0 & 2 & -4 \\ 0 & -3 & 5 & 3 \end{bmatrix}$$

Strategy: Use the EOP to get a matrix with three zeros in a row or column and use that row or column for the cofactor expansion.

Solution

Follow the Strategy. Several choices seem reasonable, including using the last 1 in the first row to get three zeros in the first row, or using the -1 in the first column to get zeros in the first column or in the third row. To get zeros in the first column, perform the following elementary row operations: $-2R_3 + R_1 \rightarrow R_1$ and $2R_3 + R_2 \rightarrow R_2$. The result is matrix B. Evaluate its determinant by the first column expansion.

$$|B| = \begin{vmatrix} 0 & 2 & -4 & 9 \\ 0 & -1 & 7 & -8 \\ -1 & 0 & 2 & -4 \\ 0 & -3 & 5 & 3 \end{vmatrix} = 0 \cdot C_{11} + 0 \cdot C_{21} + (-1)C_{31} + 0 \cdot C_{41}.$$

Thus

$$|A| = |B| = (-1)(+1) \begin{vmatrix} 2 & -4 & 9 \\ -1 & 7 & -8 \\ -3 & 5 & 3 \end{vmatrix}$$

Apply elementary row operations $2R_2 + R_1 \rightarrow R_1$ and $-3R_2 + R_3 \rightarrow R_3$ to get a matrix with two zeros in the first column:

$$|B| = (-1) \begin{vmatrix} 0 & 10 & -7 \\ -1 & 7 & -8 \\ 0 & -16 & 27 \end{vmatrix} = (-1) \begin{vmatrix} 10 & -7 \\ -16 & 27 \end{vmatrix}$$

$$= -(270 - 112) = -158.$$

Since $|A| = |B|, |A| = -158.$ ◀

Technology and Larger Determinants

The arithmetic of determinant evaluation grows so rapidly that computers and calculators must use approximation techniques. Most graphing calculators will give excellent approximations for determinants (look for operations in the Matrix menu). To use the power of this technology well, we must understand something about determinants ourselves while at the same time being alert to computational limitations.

As a simple example, we know from the definition that a determinant is a sum of signed products of entries of a matrix. It follows that *if all the entries in a matrix are integers, then its determinant must be an integer.* For

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

HISTORICAL NOTE   DETERMINANTS

Most students of mathematics today learn about determinants only in connection with matrices. Historically, though, determinants had a lively role of their own long before matrices were recognized. Matrices as such have been studied only for a little more than one hundred years, and were not widely known even into the first third of this century (see "Matrices" in Section 9.2). Determinants are numbers rather than arrays, and it probably should not be surprising that they have been recognized more than twice as long as matrices.

At least three important mathematicians independently developed and used some properties of determinants. Leibnitz, best known for his part in the invention of calculus, wrote letters in 1693 that described how to determine whether a given system of homogeneous equations is consistent by calculating a single number, which we now call a determinant. Maclaurin probably used Cramer's rule
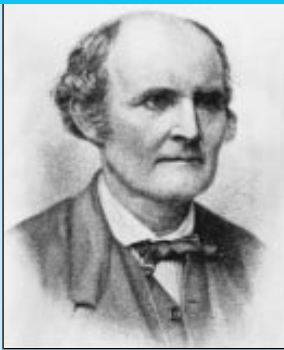
English mathematician Arthur Cayley

twenty years before Cramer published it in 1750.

We would probably not recognize Cramer's rule in its original form. It used none of the special notation we use today. There were also formulas for the solution of three by three systems, but it is likely that neither Maclaurin nor Cramer extended the rule to larger systems—with good reason. A formula for quotients of two 24-term expressions is too complicated to be worth much.

By 1773 Lagrange was using essentially modern notation for certain problems. He is responsible for the formula given in this section for the area of a triangle as a determinant. Cauchy applied the name *determinant* to a class of functions including those that we now call determinants, and Jacobi broadened Cauchy's usage to a determinant consisting of derivatives. Cayley finally related determinants and matrices in 1858, when he used them to describe points and lines in higher-dimensional geometry.

several calculators (including the TI-81 and TI-82) give $|A| = 0$, but the TI-85 returns a value of $-2.4\text{E-}12$, or $|A| = -0.0000000000024$. Obviously the TI-85 is programmed in a way that gives an approximation that is (very slightly) in error. This is not a criticism of the TI-85; every calculator will fail on some relatively simple similar example. What we need to recognize is the meaning of the result. When we see such a ridiculously small number, we should understand that the calculator is telling us (see Exercise 12) that the determinant of matrix A is equal to zero.

If you keep such calculator limitations in mind, you should not hesitate to use your calculator to check all determinant computations. The chances are very good that your calculator makes fewer arithmetic errors than you do, and the greatest source of error is probably entering numbers incorrectly or pressing a wrong key.

Why learn cofactor expansion?   With all of the power and convenience of calculator computation, why shouldn't we rely entirely on technology? In addition to the fact that we cannot use technology wisely without having some feeling for what a machine is doing for us ("garbage in, garbage out"), it turns out that a number of

the most important applications of determinants require the evaluation of highly symbolic determinants, where the result is not a number at all. In vector calculus and linear algebra and differential equations, it is necessary to know how to calculate and manipulate determinants; it is not enough to know what buttons to push to get a number.

In the next example we illustrate the use of a determinant involving unit vectors, *i, j,* and *k* that are used in physics and engineering. This particular example computes the cross product of two vectors, an operation that we do not discuss but that is used in calculus. Example 5 comes directly from linear algebra.

▶**EXAMPLE 4** *A vector product* *Looking Ahead to Calculus* Suppose $u = i + 2k$, $v = 3i - j + k$ are vectors in 3-space. Then the cross product of u and v is given by

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 3 & -1 & 1 \end{vmatrix},$$

where the second and third rows are the components of u and v. Use cofactor expansion by the first row to obtain the cross product in standard form.

Solution
Using the definition,

$$u \times v = i\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix}$$
$$= i(0 + 2) - j(1 - 6) + k(-1 - 0)$$
$$= 2i + 5j - k.$$

This last expression describes a 3-dimensional vector that is perpendicular to the two vectors u and v. ◀

▶**EXAMPLE 5** *A determinant equation* **(a)** Expand the determinant and **(b)** solve the equation for x.

$$\begin{vmatrix} x - 1 & -4 & 2 \\ -3 & x - 2 & -4 \\ 0 & 0 & x + 1 \end{vmatrix} = 0$$

Solution

(a) Using the cofactor expansion by the last row (since there are two zeros), the determinant equals

$$0 - 0 + (x + 1)\begin{vmatrix} x - 1 & -4 \\ -3 & x - 2 \end{vmatrix} = (x + 1)((x - 1)(x - 2) - 12)$$
$$= (x + 1)(x^2 - 3x - 10) = (x + 1)(x - 5)(x + 2).$$

(b) The equation reduces to $(x + 1)(x - 5)(x + 2) = 0$, whose solutions are given by $x = -2, -1, 5$. We suggest that you check by substituting each x-value into the original determinant. ◀

Applications of Determinants

As suggested in the previous examples, applications of determinants abound in different areas of mathematics. We will see another in Section 9.6 when we use inverses of matrices for solving systems of linear equations. Determinants also provide a convenient way to do some things we have previously considered in this text, among them a way of writing an equation for a line through two given points and another way to compute the area of a triangle from the coordinates of its vertices. We are not interested here in deriving Equations (2) and (3), but are merely illustrating uses of determinants. Examples and exercises support the validity of these formulas.

Equation of a line

Given two points $P(a, b)$ and $Q(c, d)$, an equation for the line PQ may be written as

$$\begin{vmatrix} 1 & x & y \\ 1 & a & b \\ 1 & c & d \end{vmatrix} = 0. \qquad (2)$$

Area of a triangle

Given $\triangle PQR$ with vertices $P(a, b)$, $Q(c, d)$, and $R(e, f)$ going around the triangle *counterclockwise,* then the area K of the triangle is given by

$$K = \frac{1}{2}\begin{vmatrix} 1 & a & b \\ 1 & c & d \\ 1 & e & f \end{vmatrix}. \qquad (3)$$

If we disregard the order of vertices, then we must take the absolute value of the determinant.

▶**EXAMPLE 6** *Determinant applications* Given points $A(-1, 1)$, $B(0, -2)$, $C(5, 3)$.

(a) Verify that Equation (2) gives an equation for the line AC.
(b) Show that $\triangle ABC$ is a right triangle and verify that the area K of the triangle is given by Equation (3).

Solution

(a) Figure 16 shows $\triangle ABC$ and line AC. Substituting the coordinates of points A and C into Equation (2) and expanding by the first row gives us

$$\begin{vmatrix} 1 & x & y \\ 1 & -1 & 1 \\ 1 & 5 & 3 \end{vmatrix} = 1(-3 - 5) - x(3 - 1) + y(5 + 1) = 0,$$

or $-x + 3y = 4$, which is obviously an equation of a line. It is a simple task to verify that the coordinates of both A and C satisfy the equation, so Equation (2) is an equation for the line containing the points A and C.

(b) From the diagram in Figure 16 we see that the slope of line AC is $\frac{1}{3}$ and the slope of line AB is -3. Thus the lines are perpendicular and $\triangle ABC$ is a right triangle. Using Equation (3), we can go around the triangle counterclockwise
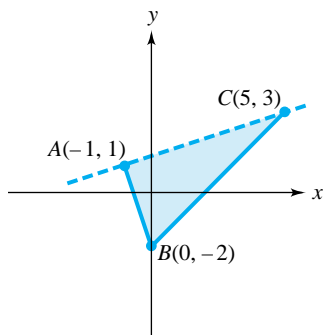

FIGURE 16

in order ABC (or, if we prefer, BCA or CAB). We have, using the first row for cofactor expansion,

$$K = \frac{1}{2} \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 5 & 3 \end{vmatrix} = \frac{1}{2}(1(0 + 10) - (-1)(3 + 2) + 1(5 - 0)) = 10.$$

Because we have a right triangle with legs b and c, we can compute the area as $\frac{1}{2}bc$ as soon as we have those lengths.

$$b = |\overline{AC}| = \sqrt{6^2 + 2^2} = 2\sqrt{10}, \quad c = |\overline{AB}| = \sqrt{1^2 + 3^2} = \sqrt{10}.$$

Thus $K = \frac{1}{2}bc = \frac{1}{2}(2\sqrt{10})(\sqrt{10}) = 10$, in agreement with Equation (3). ◄

It is interesting to observe that Equation (3) does not depend on whether or not the triangle has a right angle. Equation (3) can be used with any triangle in the coordinate plane. To find the area of a general triangle without the use of a determinant would require considerably more work.

Cramer's Rule

We conclude this section by revisiting a topic we introduced in Section 9.1. There is a technique, known as Cramer's Rule, for solving systems of linear equations using determinants. In Section 9.1 we solved a 2×2 linear system directly and observed that the solution could be expressed in terms of what we now know are determinants. The same process works for any $n \times n$ linear system. For completeness, we state the theorem here in its more general form, but we do not recommend its use for larger systems. Computationally it is too inefficient. In the next section we will get a matrix approach that is very easy to implement with technology.

Cramer's rule

Given a system of n linear equations in variables $x_1, x_2, \ldots, x_n$, where A is the coefficient matrix and B is the column of constants, let $D = |A|$ and let D_i be the determinant of the matrix obtained by replacing the ith column of A by column B. If $D \neq 0$, the system has a unique solution given by

$$x_1 = \frac{D_1}{D}, \quad ex_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}.$$

EXERCISES 9.5

Check Your Understanding

Exercises 1–6   True or False. Give reasons.

1. The determinant of

$$\begin{bmatrix} 2 & -1 \\ 3 & -5 \end{bmatrix}$$

is equal to -7.

2. The only solution of the equation

$$\begin{vmatrix} x & -2 \\ 4 & 2 \end{vmatrix} = 6$$

is given by $x = -1$.

3. $\begin{vmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}.$

4. The solution of the equation

$$\begin{vmatrix} x & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{vmatrix} = 3$$

is given by $x = -1$.

5. The solution set for the equation

$$\begin{vmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{vmatrix} = 1$$

is the empty set.

6. If every element of 2×2 matrix A is a positive number, then the determinant of A is a positive number.

Exercises 7–10 Fill in the blank so that the resulting statement is true. All questions refer to the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}.$$

7. The determinant of A is equal to _____ .

8. The minor M_{31} is equal to _____ .

9. The cofactor C_{11} is equal to _____ .

10. The cofactor C_{12} is equal to _____ .

Develop Mastery

Exercises 1–4 Cofactor Evaluation Evaluate the indicated cofactors.

1. $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -5 \\ 1 & 0 & -2 \end{vmatrix}$ Find C_{12}, C_{31}.

2. $\begin{vmatrix} -1 & 0 & 0 \\ 2 & 5 & 3 \\ 2 & -1 & 4 \end{vmatrix}$ Find C_{23}, C_{32}.

3. $\begin{vmatrix} 0 & -2 & \sqrt{3} \\ 5 & \sqrt{3} & 2 \\ -2 & 0 & 1 \end{vmatrix}$ Find C_{22}, C_{33}.

4. $\begin{vmatrix} 2 & -1 & e \\ 3 & e & -1 \\ 5 & 2 & -3 \end{vmatrix}$ Find C_{11}, C_{13}.

Exercises 5–12 Determinants by Cofactors Evaluate the determinant of the given matrix. Use cofactors for the 3×3 matrices.

5. $A = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$

6. $A = \begin{bmatrix} 0 & 4 \\ -3 & 2 \end{bmatrix}$

7. $B = \begin{bmatrix} 6 & -4 & 1 \\ 7 & -4 & 1 \\ 6 & -3 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 1 & 5 \end{bmatrix}$

9. $B = \begin{bmatrix} 2 & 5 & 2 \\ 6 & 2 & -1 \\ 2 & 2 & 1 \end{bmatrix}$

10. $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

11. $C = \begin{bmatrix} 3 & -2 & 2 \\ 4 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

12. $D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Exercises 13–20 Determinants by Technology Use a calculator to evaluate the determinant of the matrix.

13. $A = \begin{bmatrix} 2 & -1 & 0 \\ \sqrt{3} & \sqrt{12} & \sqrt{27} \\ \sqrt{75} & \sqrt{48} & -\sqrt{3} \end{bmatrix}$

14. $B = \begin{bmatrix} 1 & 3 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 2 & 0 & 4 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

15. $C = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & -1 & 2 & 1 \\ -2 & 1 & 4 & -1 \\ -1 & -2 & 3 & 1 \end{bmatrix}$

16. $A = \begin{bmatrix} -1 & 0 & -2 & 0 \\ -3 & 2 & 1 & -3 \\ 1 & -2 & -1 & 3 \\ 2 & 1 & 4 & -2 \end{bmatrix}$

17. $A = \begin{bmatrix} 0.3 & 0.7 & 1.2 \\ -0.8 & -1.3 & 0.4 \\ 0.0 & 1.0 & 2.1 \end{bmatrix}$

18. $D = \begin{bmatrix} 4 & 12 & 28 \\ 2 & -1 & 0 \\ 30 & 20 & 70 \end{bmatrix}$

19. $M = \begin{bmatrix} 1001 & 101 & 11 \\ 2001 & 201 & 21 \\ 4001 & 401 & 41 \end{bmatrix}$

20. $D = \begin{bmatrix} 17 & 0 & 0 & 0 \\ -83 & 20 & 0 & 0 \\ 25 & 100 & 500 & 0 \\ -6 & -8 & -10 & 5000 \end{bmatrix}$

Exercises 21–34 Solving for x The equation involves the variable x. (a) Expand the determinant and (b) solve for x.

21. $\begin{vmatrix} 2x & -4 \\ 3x & 2 \end{vmatrix} = 3$

22. $\begin{vmatrix} 3 & -4x \\ x & -5 \end{vmatrix} = 6$

23. $\begin{vmatrix} e^x & e \\ e & 1 \end{vmatrix} = 0$

24. $\begin{vmatrix} 1 & 0 & x \\ 3 & -1 & 2 \\ -5 & 3 & 0 \end{vmatrix} = -2$

25. $\begin{vmatrix} -2x & 1 & 0 \\ 0 & 3 & 2 \\ -x & 1 & 5 \end{vmatrix} = 4$

26. $\begin{vmatrix} x & 4 & 0 \\ 2 & 2 & -x \\ 1 & 1 & 1 \end{vmatrix} = 0$

27. $\begin{vmatrix} x & -2x & 1 \\ 3x & 1 & -2 \\ 2x & 2x+1 & -3 \end{vmatrix} = 0$

28. $\begin{vmatrix} e^x & 0 & 1 \\ -1 & 2 & -3 \\ 0 & 2 & -1 \end{vmatrix} = 4$

29. $\begin{vmatrix} 4\sin x & 1 & 0 \\ & -1 & 1 & 3 \\ & 2 & -1 & -1 \end{vmatrix} = -3$

30. $\begin{vmatrix} x & 0 & 1 \\ x & 2 & -1 \\ 1 & x & 1 \end{vmatrix} = 0$

31. $\begin{vmatrix} x-2 & 1 & 0 \\ 0 & x-3 & -1 \\ 3 & 1 & x+1 \end{vmatrix} = 0$

32. $\begin{vmatrix} x-1 & 3 & 2 \\ -2 & x-4 & -4 \\ 0 & -1 & x-2 \end{vmatrix} = 0$

33. $\begin{vmatrix} x & 0 & 0 \\ x & x+1 & 5 \\ 2 & 1 & x-1 \end{vmatrix} = 0$

34. $\begin{vmatrix} x-1 & 2 & 2 \\ 2 & x-2 & -2 \\ 1 & 0 & x \end{vmatrix} = 0$

Exercises 35–40 **Cramer's Rule** *Use Cramer's Rule to solve the system of equations. If the determinant of the coefficient matrix is zero, use Gaussian elimination.*

35. $6.3x + 2.1y = 18.9$
$1.5x + 3.4y = -4.2$

36. $2.4x - 5.2y = -8.0$
$1.6x + 2.4y = 6.4$

37. $371x + 285y = 2726$
$137x + 125y = 977$

38. $325x - 175y = -625$
$173x - 276y = 33$

39. $\quad x + y - 2z = 0$
$\quad 3x - 2y - z = 0$
$\quad -x + 4y - 3z = 0$

40. $\quad x + 2y - z = 5$
$\quad 2x + y + 2z = 3$
$\quad x - y + 3z = 0$

Exercises 41–48 **Areas of Polygons** *Find the area enclosed by the polygon with the given vertices. (Hint: If there are more than three vertices, break up the figure into triangles.)*

41. $A(1, 0)$, $B(6, 4)$, $C(8, 0)$

42. $A(1, 0)$, $B(5, -2)$, $C(7, 2)$

43. $A(2, 0)$, $B(1, -2)$, $C(4, 3)$

44. $A(5, 5)$, $B(5, -5)$, $C(0, -1)$

45. $A(1, 0)$, $B(4, 6)$, $C(8, 0)$, $D(7, -3)$

46. $A(0, 0)$, $B(4, 6)$, $C(3, 0)$, $D(5, -2)$

47. $A(0, 4)$, $B(2, 4)$, $C(0, 2)$, $D(-2, 4)$

48. $A(0, 0)$, $B(7, -3)$, $C(8, 0)$, $D(8, 6)$, $E(4, 6)$

Exercises 49–52 **Explore** *Evaluate the three determinants. State a theorem about such determinants and explain why you think your theorem is true.*

49. (a) $\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}$
(b) $\begin{vmatrix} 5 & -3 & 2 \\ 0 & 0 & 0 \\ 4 & 1 & -3 \end{vmatrix}$

(c) $\begin{vmatrix} 5 & 3 & 0 & -1 \\ 0 & 2 & 0 & 5 \\ 4 & -6 & 0 & 8 \\ 5 & 0 & 0 & -2 \end{vmatrix}$

50. (a) $\begin{vmatrix} 2a & a \\ 3 & 4 \end{vmatrix}$
(b) $\begin{vmatrix} k & 5 \\ -2k & 3 \end{vmatrix}$
(c) $\begin{vmatrix} 2 & 1 \\ c & -c \end{vmatrix}$

51. (a) $\begin{vmatrix} 1 & a & 0 \\ 2 & -a & 3 \\ -1 & 2a & 1 \end{vmatrix}$
(b) $\begin{vmatrix} k & -k & k \\ 2 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 0 & -2 \\ 5 & 1 & 0 \\ c & 2c & c \end{vmatrix}$

52. (a) $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & -5 & 1 \end{vmatrix}$
(b) $\begin{vmatrix} 2 & 6 & -3 \\ 6 & 18 & -9 \\ 1 & 0 & 5 \end{vmatrix}$

(c) $\begin{vmatrix} 40 & -25 & 0 \\ 8 & -5 & 0 \\ 5 & 1 & -4 \end{vmatrix}$

(*Hint*: Consider the first two rows.)

Exercises 53–54 **Lines Through Two Points** *Use Equation (2) to find an equation for the line that passes through points P and Q.*

53. $P(-1, 2)$ $Q(3, 4)$

54. $P(2, -3)$ $Q(-3, 5)$

Exercises 55–56 **Cross Product** *Find the cross product of the vectors **u** and **v**. See Example 4.*

55. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\quad \mathbf{v} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

56. $\mathbf{u} = \mathbf{i} - \mathbf{j}$, $\quad \mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$