Přírodovědecká fakulta
Faculty of Science
Jihočeská univerzita v Českých Budějovicích
University of South Bohemia in České Budějovice

UMB 551I Linear algebra: Tutorials

Eva Lálová, Lenka Zalabová

Contents

1. Matrices

Summary: Matrix of type $m \times n$ is a rectangular schema A with m rows and n columns

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}), \qquad \begin{matrix} i = 1, \dots, m, \\ j = 1, \dots, n, \end{matrix}'$$

where m is the number of rows and n is the number of columns.

Example 1. *Find a matrix*

(1) *of type* 4×6 *such that* $a_{ij} = i + j$
(2) *of type* 3×3 *such that* $a_{ij} = (-1)^{i \cdot j}$
(3) *of type* 2×4 *such that* $a_{ij} = (-1)^{i+j}$
(4) *of type* 4×3 *such that* $a_{ij} = i \cdot j - (i + j)$

Summary: Simple matrix operations:

- Addition - for matrices of the same type, e.g.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}.$$

- Scalar multiplication - for arbitrary matrix, e.g.

$$3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}.$$

- Transposition - for arbitrary matrix, e.g.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

Example 2. *Consider matrices*

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 1 & 4 \\ 3 & 1 & -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & -2 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 \\ 0 & -7 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -1 \\ -1 & -1 \end{pmatrix}.$$

Compute $2D - 5F$, $A + 3C$, C^T, A^T, $2C + 4E^T$, $B^T - 2G$, $D^T + F^T$, $2A + 3G$, $\ldots$.

Summary: Two matrices are equal if and only if they are of the same type, and the corresponding elements are identical.

Example 3. *For which parameters* $\alpha, \beta, \gamma, \delta$ *are the matrices* A, B *equal:*

(1) $A = \begin{pmatrix} 2\alpha + 3 & 4 & 6 & 8 \\ 8 & 12 & 6 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 10\alpha + 1 & 2\beta + 3 & 6 & 8 \\ 8 & 6\gamma + 2 & 3\delta & 4 \end{pmatrix}$

(2) $A = \begin{pmatrix} 7\alpha + 3 & 5 & 2 \\ 3\beta - 1 & 3 & 1 \\ \gamma & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 7\alpha + 3 & 5\alpha & 2 \\ 5\beta + 1 & 3\alpha & -\beta \\ \gamma & 2 & 3 \end{pmatrix}$

(3) $A = \begin{pmatrix} 2 & -1 & -\alpha & -\beta \\ 1 & 2 & \gamma & \alpha^2 \\ \gamma + 1 & \beta & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 2\alpha + \beta & \beta - \gamma \\ 1 & 2 & 3\gamma - 2 & \beta^2 \\ 1 - \beta & -\gamma & -1 & 2 \end{pmatrix}$

Summatry: Matrix is symmetric if and only if $A = A^T$. Matrix is antisymmetric if and only if $A = -A^T$.

Example 4. *Is any of matrices of example (2) symmetric or antisymmetric? Give a nontrivial example of symmetric and antisymmetric matrix of type* 3×3.

Example 5. *Solve the matrix equation* $3A + 2X = C - 2B$, *where* X *is unknown matrix and*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 6 & -3 \end{pmatrix}, C = \begin{pmatrix} 5 & 10 & 21 \\ 4 & 15 & 15 \end{pmatrix}.$$

Example 6. *Solve the matrix equation* $X + X^T - 3A = A + A^T$, *where* X *is unknown matrix and*

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}.$$

Summary: Consider matrix $A = (a_{ij})$ of type $m \times n$ and $B = (b_{ij})$ of type $n \times p$. Then there is a product $C = A \cdot B$ and it holds that

- matrix C is of type $m \times p$,
- elements of the matrix $C = (c_{ij})$ are of the form $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Thus we compute scalar (dot) product of ith row of matrix A and jth column of matrix B and we write the result into the matrix C on the position (i, j). E.g. for matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$$

$$AB = \begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}\cdot\begin{pmatrix}7 & 8\\ 9 & 10\end{pmatrix} = \begin{pmatrix}1\cdot 7+2\cdot 9 & 1\cdot 8+2\cdot 10\\ 3\cdot 7+4\cdot 9 & 3\cdot 8+4\cdot 10\\ 5\cdot 7+6\cdot 9 & 5\cdot 8+6\cdot 10\end{pmatrix} = \begin{pmatrix}7+18 & 8+20\\ 21+36 & 24+40\\ 35+54 & 40+60\end{pmatrix} = \begin{pmatrix}25 & 28\\ 57 & 64\\ 89 & 100\end{pmatrix}$$

$$BA = \begin{pmatrix}7 & 8\\ 9 & 10\end{pmatrix}\cdot\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix} = \text{not defined}$$

Example 7. *Consider matrices*

$$A=\begin{pmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{pmatrix},\quad B=\begin{pmatrix}4 & -1\\ 0 & 2\end{pmatrix},\quad C=\begin{pmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{pmatrix},$$

$$D=\begin{pmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{pmatrix},\quad E=\begin{pmatrix}1 & 0 & -2\end{pmatrix},\quad F=\begin{pmatrix}3\\ 2\\ -1\end{pmatrix}.$$

Compute BC, CB, AC, CA, AB, BA, DA, EF, FE, $\ldots$

Example 8. *Compute expression* $(A+B)^2-3E$ *for matrices*

$$A=\begin{pmatrix}-1 & 2 & 0\\ 2 & 3 & -1\\ -5 & 2 & 3\end{pmatrix},\quad B=\begin{pmatrix}3 & 1 & -3\\ 0 & -2 & 1\\ 4 & -2 & -2\end{pmatrix}.$$

Example 9. *Compute expression*$A(2E-A)^2$ *for matrix*

$$A=\begin{pmatrix}2 & -1 & 3\\ 0 & 1 & 2\\ -2 & 4 & 5\end{pmatrix}.$$

Example 10. *Solve matrix equations*

(1) $2X^T+A=BC$,
(2) $C-X=AB$,
(3) $X=AB-BA$

for

$$A=\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix},\quad B=\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix},\quad C=\begin{pmatrix}-1 & -2\\ -3 & -4\end{pmatrix}.$$

Example 11. *Solve matrix equation* $X+AA^T-B=-X^T-A^TA+3B$ *for*

$$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\quad B=\begin{pmatrix}-2 & 1\\ 1 & 2\end{pmatrix}.$$

Example 12. *Solve matrix equation* $AX=XA$ *for*

$$A=\begin{pmatrix}-2 & 1\\ 2 & 3\end{pmatrix}.$$

Example 13. *Solve matrix equation* $3X-E=AB$ *for*

$$A=\begin{pmatrix}1 & 0 & -1\\ 1 & -1 & -1\\ 2 & 0 & -2\end{pmatrix},\quad B=\begin{pmatrix}2 & 0 & -1\\ 0 & -4 & 1\\ 3 & 1 & -1\end{pmatrix}.$$

2. Vectors

Drobné opakování: Vectors are elements of $\mathbb{R}^n$, i.e. n-tuples of numbers. We define operations of addition and scalar multiplication coordinate-wise.

Example 14. *Consider vectors*

$$u = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} \in \mathbb{R}^3.$$

Find vector x such that

(1) $x = 2u + v - 3w$,
(2) $x - 2u = 3x + 2(u - v - 2w)$.

Example 15. *For which values of parameters m and n are the vectors a, b equal, where*

$$a = \begin{pmatrix} 2m \\ 3 \end{pmatrix}, b = \begin{pmatrix} 2 \\ m+n \end{pmatrix} \in \mathbb{R}^2.$$

Summary: Consider vectors $v_1, \dots, v_k$ and scalars $a_1, \dots, a_k$. Vector

$$u = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

is a linear combination of vectors v_i with coefficients a_i.

Example 16. *Write the vector x as a linear combination of vectorsu* $= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$, *where*

(1)

$$x = \begin{pmatrix} 10 \\ 9 \\ 4 \end{pmatrix},$$

(2)

$$x = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}.$$

Summary: Vectors $v_1, \dots, v_k$ are linearly independent if

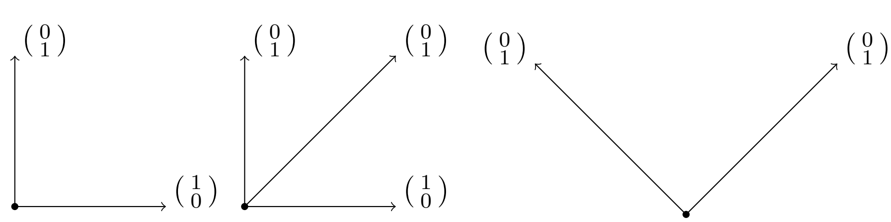
$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = o \Rightarrow a_1 = a_2 = \cdots = a_k = 0.$$

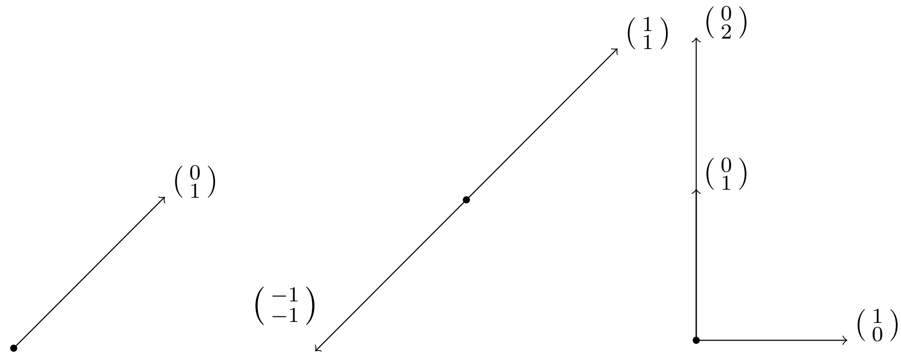
Thus the only way how to write zero vector as a linear combination of vectors v_i is to take zero vector. They are linearly dependent in the opposite situation

$$\exists a_1, \dots, a_k : a_1 v_1 + \cdots + a_k v_k = o \wedge \exists i : a_i \neq 0.$$

Thus there is a way how to write zero vector as non-zero linear combination of vectors v_i.

Example 17. *Are the following vectors dependent or independent?*





Example 18. *Compute from the definition if the vectors from $\mathbb{R}^n$ are dependent or independent.*

(1)
$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1\\-4\end{pmatrix}, \begin{pmatrix}-1\\5\end{pmatrix} \in \mathbb{R}^2$$

(2)
$$\begin{pmatrix}3\\-1\\5\end{pmatrix}, \begin{pmatrix}0\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\2\end{pmatrix} \in \mathbb{R}^3$$

(3)
$$\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}2\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\3\end{pmatrix} \in \mathbb{R}^4$$

(4)
$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix} \in \mathbb{R}^3$$

3. Row echelon form of matrices and their rank

Summary: The matrix is in a row echelon form if all nonzero rows are above rows of all zeroes and the leading coefficient of a non–zero row is always strictly to the right of the leading coefficient of the row above it. E.g. matrices

$$\begin{pmatrix}0&0&1\\0&0&2\\3&4&5\end{pmatrix}, \quad \begin{pmatrix}1&2&3\\0&4&5\\0&7&8\end{pmatrix}$$

are not in row echelon form while

$$\begin{pmatrix}1&3&4\\0&0&3\\0&0&0\end{pmatrix}$$

is in the row echelon form. How to find row echelon form? Elementary row transformations:

- switching of two rows,
- multiplication of a given row by a non–zero number,
- adding a row to another row.

E.g. for the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

we get

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 5 & -5 \\ 0 & 5 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 5 & -5 \\ 0 & 0 & -2 \end{pmatrix}$$

Example 19. *Find row echelon form of matrices.*

$$B = \begin{pmatrix} 1 & 8 & 3 \\ -2 & 3 & -1 \\ -1 & 11 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 2 & -2 \\ 0 & 2 & 5 & 2 \\ -1 & 2 & 7 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 3 & 1 & 0 \\ 3 & 3 & 1 & -1 \\ 1 & 2 & 1 & 2 \end{pmatrix}.$$

Summary: A rank of the matrix is the (maximal) number of linearly independent rows. How to find the rank?

- Find the row echelon form.
- The rank of the matrix in the row echelon form equals to the number of its nonzero rows.

Example 20. *Find the row echelon form of matrices and their rank.*

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 3 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & 0 & -3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -3 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 & 3 & -2 & 4 \\ 4 & -2 & 5 & 1 & 7 \\ 2 & -1 & 1 & 8 & 2 \end{pmatrix}$$

Example 21. *Find the rank of matrices.*

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -3 & 5 \\ 4 & -12 & 20 \\ 3 & -9 & 15 \\ 2 & -6 & 10 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 5 & -2 \\ 4 & -3 & 1 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & -1 & 2 \\ 3 & -4 & 4 \\ 8 & 2 & 1 \\ 3 & 5 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \\ 3 & 2 & 4 \\ 0 & 5 & -5 \end{pmatrix}$$

Example 22. *Find the rank of matrix depending on parameter x.*

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & -2 & 18 \\ x & 2 & 1 & 5 \\ 3 & 1 & 2 & -2 \end{pmatrix}$$

Summary: How to decide about the linear (in)dependence of a system of vectors in $\mathbb{R}^n$:

- Each system of $n+1$ (or more) vectors is linearly dependent.
- If we get a system of n (or less) vectors, then construct a matrix consisting of these vectors.

– If the rank equals to the number of vectors, then the system is independent.
– If the rank is strictly smaller, then the system is dependent.

Example 23. *Compute if the vectors are dependent or independent.*

(1) $\begin{pmatrix}1\\2\\-2\end{pmatrix}, \begin{pmatrix}-2\\3\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\2\end{pmatrix} \in \mathbb{R}^3$

(2) $\begin{pmatrix}1\\2\\-1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\3\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\2\end{pmatrix} \in \mathbb{R}^3$

(3) $\begin{pmatrix}1\\1\\-1\\2\end{pmatrix}, \begin{pmatrix}-4\\1\\1\\-3\end{pmatrix}, \begin{pmatrix}2\\-3\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \in \mathbb{R}^4$

Example 24. *Compute if the columns of matrices are dependent or independent.*

$$\begin{pmatrix}1&-2&-1\\2&-3&2\\-2&1&2\end{pmatrix}, \quad \begin{pmatrix}1&1&-1\\3&1&2\\-2&2&-8\end{pmatrix}, \quad \begin{pmatrix}0&2&1\\-1&1&0\\2&-1&1\\3&-2&1\end{pmatrix}, \quad \begin{pmatrix}-1&1&0\\0&2&1\\0&1&1\\0&1&1\end{pmatrix}.$$

4. Systems of linear equations

Summary: The system $Ax = b$ has a solution if and only if $r(A) = r(A|b)$. Steps in the row echelon form suggest where to put parameters. E.g. for the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ x_1 - x_2 - x_3 &= 1 \\ 3x_1 - 5x_2 + x_3 &= 5 \end{aligned}$$

we get

$$\left(\begin{array}{ccc|c}1&-2&1&2\\1&-1&-1&1\\3&-5&1&5\end{array}\right) \sim \left(\begin{array}{ccc|c}1&-2&1&2\\0&1&-2&-1\\0&1&-2&-1\end{array}\right) \sim \left(\begin{array}{ccc|c}1&-2&1&2\\0&1&-2&-1\\0&0&0&0\end{array}\right)$$

$$x_3 = t, \quad x_2 = 2t - 1, \quad x_1 = 2 - t + 4t - 2 = 3t$$

$$x \in \left\{ \begin{pmatrix}0\\-1\\0\end{pmatrix} + t\begin{pmatrix}3\\2\\1\end{pmatrix} : t \in \mathbb{R} \right\}$$

Example 25. *Solve the system using Gaussian elimination method. Write results into vector form.*

(1)

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_1 - 2x_2 + x_3 &= 0 \\ 5x_1 - 9x_2 + 5x_3 &= 1 \end{aligned}$$

(2)

$$\begin{aligned} 0.2x_1 - 3.8x_2 &= 4.3 \\ 0.4x_1 - 7.6x_2 &= 3.2 \end{aligned}$$

(3)

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 4 \\ x_2 - x_3 + x_4 &= -3 \\ 2x_1 + 6x_2 - 6x_4 &= 2 \\ -7x_2 + 3x_3 + x_4 &= -3 \end{aligned}$$

(4)

$$\begin{aligned} 6x_1 - 9x_2 + 7x_3 + 10x_4 &= 3 \\ 2x_1 - 3x_2 - 3x_3 - 4x_4 &= 1 \\ 2x_1 - 3x_2 + 13x_3 + 18x_4 &= 1 \end{aligned}$$

(5)

$$\begin{aligned} 3x_1 + 3x_2 - 4x_3 + 4x_4 &= 0 \\ 2x_1 + x_2 - 2x_3 + x_4 &= 0 \\ x_1 - x_2 - 2x_4 &= 0 \\ 6x_1 + 6x_2 - 8x_3 + 8x_4 &= 0 \end{aligned}$$

(6)

$$\begin{aligned} 3x_1 + 2x_5 &= 1 \\ x_1 + x_5 &= 1 \\ 7x_1 + 4x_2 &= 1 \\ 5x_1 + 3x_5 &= 1 \end{aligned}$$

Example 26. *Solve the system using Gaussian elimination method depending on a.*

(1)

$$\begin{aligned} ax_1 + x_2 + x_3 &= 1 \\ x_1 + ax_2 + x_3 &= 1 \\ x_1 + x_2 + ax_3 &= 1 \end{aligned}$$

(2)

$$\begin{aligned} x_1 - 4x_2 + 3x_3 &= 2 \\ ax_1 + x_2 - 9x_3 &= -17 \\ x_1 - 2x_2 + x_3 &= 0 \end{aligned}$$

5. Determinants

Summary: $|A| = \sum_{(j_1,\ldots,j_n)} (-1)^{\sigma(j_1,\ldots,j_n)} a_{1j_1} a_{2j_2} \ldots a_{nj_n}$.

Example 27. *Find determinants using rule of Saarus.*

$$\begin{vmatrix} 5 & 2 \\ 6 & 4 \end{vmatrix}, \quad \begin{vmatrix} 3 & -2 & 5 \\ 1 & 4 & 1 \\ 2 & -3 & 4 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix}.$$

Example 28. *Solve the equation* $\begin{vmatrix} 1 & x-2 \\ 2 & x-1 \end{vmatrix} = 0$.

Summary: Finding determinant using elementary row transformations:

- Find row echelon form,
- then determinant equals to the product of all elements on the main diagonal.

Elementary row transformations change determinants.

- zero row $\Rightarrow$ det $= 0$
- switching of arbitrary two rows $\Rightarrow$ sign change
- scalar multiplication of a row $\Rightarrow$ scalar multiplication of the determinant
- adding of a linear combination of row to another row $\Rightarrow$ nothing

E.g.

$$\begin{vmatrix} 2 & -3 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 1 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 1 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & -3 & 1 & -2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} =$$

$$-1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & -6 & 2 & -4 \\ 0 & 6 & 9 & 3 \\ 0 & 0 & -1 & 0 \end{vmatrix} = -1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & -6 & 2 & -4 \\ 0 & 0 & 11 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix} =$$

$$-1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot (-1) \cdot \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & -6 & 2 & -4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 11 & -1 \end{vmatrix} = -1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot (-1) \cdot \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & -6 & 2 & -4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} =$$

$$\frac{1}{6} \cdot 1 \cdot (-6) \cdot (-1) \cdot (-1) = 1$$

Summary: Finding determinant using rule of Laplace:

$$\begin{aligned} |A| &= \sum_{k=1}^{n} a_{ik} D_{ik} \ldots \text{expansion for ith row,} \\ |A| &= \sum_{k=1}^{n} a_{kj} D_{kj} \ldots \text{expansion for jth column,} \end{aligned}$$

where $D_{ij} = (-1)^{i+j} \cdot \big|\text{matrix of order } n-1 \text{ that we get by removing ith row and jtj column}\big|$.
E.g.

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 0 & 3 & 4 \\ 0 & 1 & 1 & 1 \end{vmatrix} = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} + 0 \cdot (-1)^{2+1} \begin{vmatrix} 3 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} + 3 \cdot (-1)^{3+1} \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$+ 0 \cdot (-1)^{4+1} \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 3 \\ 0 & 3 & 4 \end{vmatrix} = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} + 0 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

$$+ 3 \cdot 3 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + 3 \cdot 0 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + 3 \cdot 2 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= 3 - 4 + 8 - 9 + 9 \cdot (2-3) + 6 \cdot (1-2) = -17$$

This method is useful for matrices with many zeros. The best possibility is combining of both methods. E.g.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & -2 & 3 & -1 \\ 4 & 1 & 4 & 9 & 1 \\ 8 & 1 & -8 & 27 & -1 \\ 16 & 1 & 16 & 81 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -3 & 2 & -2 \\ 3 & 0 & 3 & 8 & 0 \\ 7 & 0 & -9 & 26 & -2 \\ 15 & 0 & 15 & 80 & 0 \end{vmatrix} = 1 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & -3 & 2 & -2 \\ 3 & 3 & 8 & 0 \\ 7 & -9 & 26 & -2 \\ 15 & 15 & 80 & 0 \end{vmatrix} =$$

$$-1 \cdot \begin{vmatrix} 1 & -3 & 2 & -2 \\ 3 & 3 & 8 & 0 \\ 6 & -6 & 24 & -0 \\ 15 & 15 & 80 & 0 \end{vmatrix} = -1 \cdot (-2) \cdot (-1)^{1+4} \cdot \begin{vmatrix} 3 & 3 & 8 \\ 6 & -6 & 24 \\ 15 & 15 & 80 \end{vmatrix} = -2 \cdot 6 \cdot 5 \cdot \begin{vmatrix} 3 & 3 & 8 \\ 1 & -1 & 4 \\ 3 & 3 & 16 \end{vmatrix} =$$

$$60 \cdot \begin{vmatrix} 1 & -1 & 4 \\ 3 & 3 & 8 \\ 3 & 3 & 16 \end{vmatrix} = 60 \cdot \begin{vmatrix} 1 & -1 & 4 \\ 0 & 6 & -4 \\ 0 & 6 & 4 \end{vmatrix} = 60 \cdot \begin{vmatrix} 1 & -1 & 4 \\ 0 & 6 & -4 \\ 0 & 0 & 8 \end{vmatrix} = 60 \cdot 1 \cdot 6 \cdot 8 = 360 \cdot 8 = 2880$$

Example 29. *Fond determinants.*

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}, \quad \begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 2 & 3 \\ -1 & 1 & -3 & 4 \\ -1 & -1 & 0 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}, \quad \begin{vmatrix} 3 & 2 & 1 & 5 \\ 2 & 7 & 5 & 6 \\ 6 & 4 & 1 & 0 \\ 4 & -3 & 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 0 & 2 & -1 \\ 0 & 1 & 3 & -1 & 0 \\ 1 & 2 & 0 & 2 & 1 \\ 0 & -1 & 3 & 1 & 0 \\ -1 & 2 & 0 & 2 & 1 \end{vmatrix}, \quad \begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix},$$

$$\begin{vmatrix} 1 & 0 & 2 & 0 & 3 & 0 \\ 5 & 1 & 4 & 2 & 7 & 3 \\ 1 & 0 & 4 & 0 & 9 & 0 \\ 8 & 1 & 5 & 3 & 7 & 6 \\ 9 & 1 & 5 & 4 & 3 & 8 \\ 1 & 0 & 7 & 0 & 9 & 0 \end{vmatrix}, \quad \begin{vmatrix} 7 & 6 & 9 & 4 & -4 \\ 1 & 0 & -2 & 6 & 6 \\ 7 & 8 & 9 & -1 & -6 \\ 1 & -1 & -2 & 4 & 5 \\ -7 & 0 & -9 & 2 & -2 \end{vmatrix}.$$

6. Inverse Matrices

Summary: Inverse matrix to the matrix A is a matrix A^{-1} such that $A \cdot A^{-1} = A^{-1} \cdot A = E$. Inverse matrix to the matrix A exists if A is

- square,
- regular, i.e. full rank.

Then A^{-1} is given uniquely. How to find inverse matrix?

- Write the scheme $(A|E)$,
- use row transformations to transform A to identity matrix and do the same with E,
- get $(E|A^{-1})$.

E.g. for matrix

$$A=\begin{pmatrix}1 & -1 & 1\\ 2 & 2 & 0\\ 0 & 3 & -1\end{pmatrix}$$

we get

$$\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 2 & 2 & 0 & 0 & 1 & 0\\ 0 & 3 & -1 & 0 & 0 & 1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & 4 & -2 & -2 & 1 & 0\\ 0 & 3 & -1 & 0 & 0 & 1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & 0\\ 0 & 3 & -1 & 0 & 0 & 1\end{array}\right)\sim$$

$$\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & 0\\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{4} & 1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 1\\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{4} & 1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 1\\ 0 & 0 & 1 & 3 & -\frac{3}{2} & 2\end{array}\right)\sim$$

$$\left(\begin{array}{ccc|ccc}1 & -1 & 0 & -2 & \frac{3}{2} & -2\\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 1\\ 0 & 0 & 1 & 3 & -\frac{3}{2} & 2\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 1 & -1\\ 0 & 1 & 0 & -1 & -\frac{1}{2} & 1\\ 0 & 0 & 1 & 3 & -\frac{3}{2} & 2\end{array}\right)$$

and thus

$$A^{-1}=\begin{pmatrix}-1 & 1 & -1\\ -1 & -\frac{1}{2} & 1\\ 3 & -\frac{3}{2} & 2\end{pmatrix}.$$

Example 30. *Find inverse matrices.*

$$B=\begin{pmatrix}1 & 1 & 1\\ 2 & 3 & 3\\ -1 & -3 & -2\end{pmatrix},\quad C=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix},\quad D=\begin{pmatrix}1 & 0 & 1\\ 2 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}.$$

Example 31. *Solve the matrix equation $AX=B$ for*

$$A=\begin{pmatrix}2 & -4\\ 2 & -7\end{pmatrix},\quad B=\begin{pmatrix}6 & 0\\ 18 & -6\end{pmatrix}.$$

Example 32. *Solve the matrix equation $AX+2X=X+B$ for*

$$A=\begin{pmatrix}0 & 1 & -1\\ -2 & 2 & -2\\ -2 & 0 & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & 2 & -2\\ -5 & 1 & 1\\ -2 & -2 & 4\end{pmatrix}.$$

Example 33. *Solve the matrix equation $AX-X=A^2+E$ pro*

$$A=\begin{pmatrix}2 & -1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 3\end{pmatrix}.$$

Example 34. *Solve the matrix equation $AXB=C$ for*

$$A=\begin{pmatrix}3 & -2\\ -5 & 3\end{pmatrix},\quad B=\begin{pmatrix}3 & 1\\ 2 & 2\end{pmatrix},\quad C=\begin{pmatrix}-7 & 1\\ 5 & 2\end{pmatrix}.$$

7. Eigenvalues and eigenvectirs of matrices

Summary: For a square matrix A, the eigenvector for an eigenvalue λ is such a non-zero vector v for which $Av=\lambda\cdot v$. How to find them?

- Find λ such that the matrix $|A-\lambda\cdot E|$ is singular,
- solve the linear systems $(A-\lambda\cdot E)v=o$ for corresponding eigenvalues λ.

E.g. for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we get

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0 \Rightarrow \text{eigenvalues are } 1, -1$$

$$\lambda_1 = -1 \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad x_1 = -x_2, \quad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow V_{-1} = \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$\lambda_2 = 1 \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad x_1 = x_2, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow V_1 = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Example 35. *Find eigenvalues and corresponding eigenvectors of the following matrices. (In the case you use the right method for computation of the determinant in question, you shall be able to find all eigenvalues using secondary school methods.)*

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 4 & 7 & -5 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 7 & 6 \\ 2 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & -5 \\ 6 & 4 & -9 \\ 5 & 3 & -7 \end{pmatrix}.$$