UMB 551I Linear algebra

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We follow the Czech version (UMB 551 Lineární algebra) by Jan Eisner available on fix.prf.jcu.cz/~eisner/lock/UMB-551/

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Obsah

1. Arithmetic Vector Spaces
2. Linear Dependence and Independence
3. Basis of Arithmetic Vector Space

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3. Basis of Arithmetic Vector Space

We consider real vectors, i.e. elements of $\mathbb{R}^n$. A *vector* simply is an ordered n–tuple of real numbers of the form $v = (a_1, a_2, \dots, a_n)$. The number n is called a *dimension*.

Example

$n = 2$; vectors in $\mathbb{R}^2$ are tuples of the form $v = (a_1, a_2)$. This corresponds to the usual pictures from school. Let us sketch vectors $v_1 = (1, 3)$ and $v_2 = (-2, 1)$.

(picture)

We define operations of addition and scalar multiplication coordinate-wise.

Definition

Let $u = (a_1, a_2 \ldots, a_n)$ and $v = (b_1, b_2 \ldots, b_n)$ be vectors and $r \in \mathbb{R}$. We define:

$$\begin{aligned} u + v &= (a_1, a_2 \ldots, a_n) + (b_1, b_2 \ldots, b_n) \\ &= (a_1 + b_1, a_2 + b_2 \ldots, a_n + b_n) \\ ru &= r \cdot (a_1, a_2 \ldots, a_n) = (ra_1, ra_2 \ldots, ra_n) \end{aligned}$$

Theorem

Let u, v, w be arbitrary vectors. Then the following holds:

1. $(u + v) + w = u + (v + w)$
2. $u + v = v + u$
3. *there exists a vector o such that $u + o = u$ for all vectors u: the so–called zero vector $o = (0, 0, \dots, 0)$.*
4. *For each vector v, there is the vector $-v$ such that $v + (-v) = o$, the so–called opposite vector $-v = (-a_1, -a_2, \dots, -a_n)$.*

Clearly:

$v + o = (v_1, v_2, \dots, v_n) + (0, 0, \dots, 0) =$
$(v_1 + 0, v_2 + 0, \dots, v_n + 0) = (v_1, v_2, \dots, v_n) = v$
$v + (-v) = (v_1, v_2, \dots, v_n) + (-v_1, -v_2, \dots, -v_n) =$
$(v_1 - v_1, v_2 - v_2, \dots, v_n - v_n) = (0, 0 \dots, 0) = o$

Theorem

Let u, v be arbitrary vectors and r, s scalars (real numbers). Then the following holds:

5. $r(u+v) = ru + rv$
6. $(r+s)u = ru + su$
7. $r \cdot (s \cdot u) = (rs) \cdot u$
8. $1 \cdot v = v$

The crucial observation is that all remaining identities for vectors and the two operations can be derived from the above eight identities.

Definition

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Example

$\mathbb{R}^2$ together with the "graphic operations" is two–dimensional arithmetic vector spaces.

1 Arithmetic Vector Spaces

2 Linear Dependence and Independence

3 Basis of Arithmetic Vector Space

Definition

The vector $v = r_1 v_1 + r_2 v_2 + \cdots + r_k v_k \in V_n$ is called a *linear combination of vectors* $v_1, \ldots, v_k \in V_n$ with coefficients $r_1, \ldots, r_k \in \mathbb{R}$.

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Example

In $\mathbb{R}^2$, suppose we have $v_1 = (1, 3)$, $v_2 = (3, -1)$, $v_3 = (0, 7)$. Then $v = (-2, 18)$ is a linear combination of vectors v_1, v_2, v_3 with coefficients $1, -1, 2$, because

$$1 \cdot (1, 3) + (-1) \cdot (3, -1) + 2 \cdot (0, 7) =$$
$$(1, 3) - (3, -1) + (0, 14) = (-2, 18).$$

Definition

- The system of vectors $v_1, \ldots, v_k$ is called *linearly dependent*, if there exist coefficients $r_1, \ldots, r_k$ such that
  - at least one of the coefficients is non–zero,
  - $r_1 v_1 + r_2 v_2 + \cdots + r_k v_k = o$ (the zero vector)
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- We call them *linearly independent* in the opposite situation.
- In other words, vectors $v_1, \ldots, v_k$ are linearly independent, if the equality $r_1 v_1 + r_2 v_2 + \cdots + r_k v_k = o$ implies $r_1 = r_2 = \cdots = r_k = 0$.
  (We can get the zero vector only as zero combination.)

Example

Determine if the vectors u, v, w from $\mathbb{R}^4$ are linearly dependent or independent. Here:

$$u = (2, 1, 0, 3),$$

$$v = (-1, 1, 1, 2),$$

$$w = (1, 0, -4, 1).$$

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We discuss the linear combination

$$a(2, 1, 0, 3) + b(-1, 1, 1, 2) + c(1, 0, -4, 1) = (0, 0, 0, 0),$$

where a, b, c are unknowns.

Example

Due to the coordinate–wise definition of the operations, we get the following easy linear system:

$$\begin{array}{llll} 2a & -b & +c & = 0 \\ a & +b & & = 0 \\ & b & -4c & = 0 \\ 3a & +2b & +c & = 0 \end{array}$$

We are interested in solutions of this system. More precisely, we would like to know, whether there is only the zero solution, or whether there are also some others.

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The system has only the zero solution $\Rightarrow$
the only possible coefficients are zeros $\Rightarrow$
vectors are linearly independent.

Example

Determine if the vectors u, v, w from $\mathbb{R}^3$ are linearly dependent or independent. Here:

$$u = (1, 1, 0),$$

$$v = (0, 1, 2),$$

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The system has infinite amount of solutions of the form $(-2t, -t, t)$ for arbitrary real number t.

Example

Choose $t = 1$. Then we get the solution $(-2, -1, 1)$. We get

$$-2(1,1,0) - 1(0,1,2) + 1(2,3,2) =$$
$$(-2,-2,0) + (0,-1,-2) + (2,3,2) = (0,0,0),$$

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and vectors are linearly dependent.

Observations:

- To solve the problem about the linear (in)dependency of a system of vectors, we need to solve a linear system. However, we need more effective methods to solve them that the substitution method.
- In the second example: Note that we can write one of the vectors u, v, w as a linear combination of the others.

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- To solve the problem about the linear (in)dependency of a system of vectors, we need to solve a linear system. However, we need more effective methods to solve them that the substitution method.
- In the second example: Note that we can write one of the vectors u, v, w as a linear combination of the others. We have

$$2(1,1,0)+1(0,1,2)=(2,3,2).$$

This is not possible in the case of independent vectors. (Try to discuss the first example.)

Theorem

- *Let $k \geq 2$ (i.e. we have at least two vectors). The vectors $v_1, \ldots, v_k$ are linearly dependent if and only if the exists the index i such that the vector v_i is the linear combination of the remaining vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$.*
- *Let $k = 1$. The vector v_1 is linearly dependent if and only if $v_1 = o$.*

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- *Let $k = 1$. The vector v_1 is linearly dependent if and only if $v_1 = o$.*

Corollary

- *If one of the vectors $v_1, \ldots, v_k$ is the zero vector, then the system of vectors is linearly dependent.*
- *If there exist different indices i, j such that $v_i = v_j$, then the system of vectors is linearly dependent.*

1 Arithmetic Vector Spaces

2 Linear Dependence and Independence

3 Basis of Arithmetic Vector Space

Definition

The system of vectors $(u, v, \ldots, w)$ from the arithmetic vector space is called *a basis*, if the following conditions hold:

1. it is linearly independent,
2. each vector from the arithmetic vector space can be written as their linear combination.

Example

Form the system of vectors $((1,0),(0,1))$ a basis of $\mathbb{R}^2$?
(picture)

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We verify the two conditions from the definition:

(1) The vectors are linearly independent, because we have $a(1,0)+b(0,1)=(0,0)$, and thus

$$\begin{array}{r} 1\cdot a+0\cdot b=0 \\ 0\cdot a+1\cdot b=0 \end{array} \Rightarrow a=b=0.$$

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(2) Each vector (p,q) can be written as

$$(p,q)=p(1,0)+q(0,1).$$

All together, they form a basis.

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We verify the two conditions from the definition:
(1) The vectors are linearly independent, because we have $a(1,1)+b(1,-1)=(0,0)$ and thus

$$\begin{aligned} a+b&=0\\ a-b&=0 \end{aligned} \Rightarrow a=b=0.$$

Example

(2) We would like to write (p,q) as a combination of vectors $(1,1),(1,-1)$. Let us start to find the coefficients of such combination. We have $a(1,1)+b(1,-1)=(p,q)$ and thus

$$\begin{aligned} a+b&=p \\ a-b&=q \end{aligned},$$

which is the system with two unknowns a,b and two parameters p,q.

Example

We have $a = q + b$ from the second equation and we substitute it into the first one:

$$\begin{aligned} q + b + b &= p \\ 2b &= p - q \\ b &= \tfrac{p-q}{2}. \end{aligned}$$

We get $a = q + \dfrac{p-q}{2} = \dfrac{2q+p-q}{2} = \dfrac{p+q}{2}$.
This makes sense for arbitrary p, q, i.e. we can find coefficients for arbitrary vector (p, q).
(picture)

Example

All together,

$$(p, q) = \frac{p+q}{2}(1, 1) + \frac{p-q}{2}(1, -1).$$

All together, they form a basis.

E.g., the vector $(p, q) = (3, 7)$ has coefficients

$$a = \frac{p+q}{2} = \frac{3+7}{2} = 5, \qquad b = \frac{p-q}{2} = \frac{3-7}{2} = -2$$

and we have

$$5(1, 1) - 2(1, -1) = (3, 7).$$

Example

Form the system of vectors $((1, 0), (0, 1), (1, 1))$ a basis of $\mathbb{R}^2$? (picture)

Example

Form the system of vectors $((1,0),(0,1),(1,1))$ a basis of $\mathbb{R}^2$? (picture)

We verify the two conditions from the definition:

(1) Vectors are linearly dependent, because e.g. the vector $(1,1)$ can be written as a linear combination of vectors $(1,0)$ a $(0,1)$. Clearly,

$$(1,1) = 1 \cdot (1,0) + 1 \cdot (0,1)$$

and thus

$$1 \cdot (1,0) + 1 \cdot (0,1) - 1 \cdot (1,1) = (0,0).$$

Thus they cannot form a basis.

Example

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(1)The system is linerly independent, because it consists of a one non–zero vector.

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We verify the two conditions from the definition:
(1)The system is linerly independent, because it consists of a one non–zero vector.
(2) There exists vectors, which cannot be written as linear combinations (multiples in this case) of the vector.
E. g., there is no a such that $a \cdot (1,1) = (2,3)$.
Thus it cannot form a basis.

Observations:

- The basis is the smallest possible set of vectors such that each vector can be described (generate, combined) via this set.
  It often means that it suffices to work with the basis instead of the whole arithmetic vector space.

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- The basis is the smallest possible set of vectors such that each vector can be described (generate, combined) via this set.
  It often means that it suffices to work with the basis instead of the whole arithmetic vector space.
- All basis of $\mathbb{R}^2$ (from the above examples) have 2 elements. (1 vector is not sufficient and 3 vector are too much.)
- There can exist many different basis. (Find some other basis of $\mathbb{R}^2$.)

Theorem

- *Each arithmetic vector space* $\mathbb{R}^n$ *has a basis.*
- *There always is the standard (canonical) basis:*

$$\begin{aligned} e_1 &= (1, 0, \ldots, 0), \\ e_2 &= (0, 1, \ldots, 0), \\ &\vdots \\ e_n &= (0, \ldots, 0, 1). \end{aligned}$$

- *Clearly, this system of vectors is linearly independent and each vector can be written as a combination of these vectors.*
- *Each basis of* $\mathbb{R}^n$ *has exactly* n *vectors.*

This does not mean, that each system of n vectors forms a basis!

Theorem

The system of vectors from $\mathbb{R}^n$ forms a basis of $\mathbb{R}^n$ if and only if each vector can be **uniquely** *written as a linear combination of vectors from the system.*

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Example

In $\mathbb{R}^2$, consider the canonical basis $((1,0),(0,1))$ and a vector $(3,7)$. This vector can be written as $3(1,0)+7(0,1)$ and there is no other possibility.

Example

In $\mathbb{R}^2$, consider the system $((1,0),(0,1),(1,1))$, which does not form a basis. The vector $(3,7)$ can be written as the combination

$$(3,7) = 3(1,0) + 7(0,1) + 0(1,1)$$

as well as

$$(3,7) = 2(1,0) + 6(0,1) + 1(1,1)$$

and there in infinite amount of such possibilities.

We will see later, how to solve and discuss problems concerning linear (in)dependence and basis bases more effectively using matrices.