UMB 551I Linear algebra

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We follow the Czech version (UMB 551 Lineární algebra) by Jan Eisner available on fix.prf.jcu.cz/~eisner/lock/UMB-551/

1. listopadu 2016

Obsah

Let $A = (a_{ij})$ be a (real) matrix of type $m \times n$. Then its rows are vectors and we discuss the question of their linear dependence and independence.

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A *rank* $r(A)$ of the matrix A is the (maximal) number of linearly independent rows in A.

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Definition

A *rank* $r(A)$ of the matrix A is the (maximal) number of linearly independent rows in A.

- The rank of the matrix is 0 if and only if it is the zero matrix.
- The rank of a non–zero matrix is an integer.

Theorem

- *The rank of the matrix equals to the maximal number of its linearly independent columns.*
- $r(A) = r(A^T)$

- If A is of type $m \times n$, then $r(A) \leq \min\{m, n\}$.

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Theorem

Let A be a square matrix of order n. The the following claims are equivalent:

- $r(A) = n$,
- *rows of A are linearly independent,*
- *columns of A are linearly independent.*

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Let A be a square matrix of order n. The the following claims are equivalent:

- $r(A) < n$,
- *rows of A are linearly dependent,*
- *columns of A are linearly dependent.*

Definition

Let A be a matrix of type $m \times n$. The following 'operations' are called *elementary row transformation* of A:

- switching of two rows,
- multiplication of a given row by a non–zero number,
- adding a row to another row.

Definition

Let A be a matrix of type $m \times n$. The following ‘operations’ are called *elementary row transformation* of A:

- switching of two rows,
- multiplication of a given row by a non–zero number,
- adding a row to another row.

Theorem

Elementary row transformations do not change the rank of the matrix.

We use elementary row transformations to transform the matrix into a different matrix (of the same rank) such that it is easy to see the rank from the new matrix.

Definition

Let A be a matrix of type $m \times n$. We say that the matrix is in a *row echelon form*, if:

- all nonzero rows (rows with at least one non–zero element) are above rows of all zeroes, (and thus all zero rows, if any, belong at the bottom of the matrix),
- the leading coefficient (the first non–zero number from the left, also called the *pivot*) of a non–zero row is always strictly to the right of the leading coefficient of the row above it.

Example

Is the following matrix in the row echelon form?

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 3 & 4 & 7 \\ 3 & 9 & -5 & 8 \end{pmatrix}$$

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... yes

If the matrix is in the row echelon form, then it is possible to indicate a broken line segment distinguishing the zeros downstairs and the non–zero part upstairs such that there is exactly one step in each row.

How to find the rank of the matrix?

Theorem

A non–zero matrix of arbitrary type can be transformed using finitely many elementary row transformations into the row echelon form.

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Theorem

The rank of the matrix in the row echelon form equals to the number of its non–zero rows.

Example

Find the rank of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & -1 & -2 \\ -1 & 1 & 3 & 0 & 0 & -4 \\ 2 & 1 & -3 & 0 & -1 & 2 \end{pmatrix}$$

.

Example

We would like to transform the matrix into the echelon form. Firstly, we need a non–zero number on the position $(1,1)$ and zeros below. Thus we have to switch the rows.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & -1 & -2 \\ -1 & 1 & 3 & 0 & 0 & -4 \\ 2 & 1 & -3 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 3 & 0 & 0 & -4 \\ 2 & 1 & -3 & 0 & -1 & 2 \end{pmatrix}$$

Example

Now, we can get zeros below the position $(1,1)$

$$\begin{pmatrix} 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 3 & 0 & 0 & -4 \\ 2 & 1 & -3 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 3 & 3 & 0 & -1 & -6 \\ 0 & -3 & -3 & 0 & 1 & 6 \end{pmatrix}$$

Here we make the following transformations: 3rd row $+$1st row, 4th $+(-2)\cdot$ 1st row

Example

Now, we need a non–zero number on the position $(2, 2)$ and zeros below.

$$\begin{pmatrix} 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 3 & 3 & 0 & -1 & -6 \\ 0 & -3 & -3 & 0 & 1 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -4 & -9 \\ 0 & 0 & 0 & 0 & -4 & -9 \end{pmatrix}$$

Here we make the following transformations: 3rd row $+(-3) * $ 2nd row, 4th row $+3 *$ 2nd row.

Example

Now, we need a non–zero number on the position $(3, 5)$ and zeros below. (There are zeros on the position $(3, 3)$ and below, as well as $(3, 4)$ and below.)

$$\begin{pmatrix} 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -4 & -9 \\ 0 & 0 & 0 & 0 & -4 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -4 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and this matrix is in the row echelon form. Thus the rank of the matrix A is $r(A) = 3$.

How to decide about the linear (in)dependence of a system of vectors in n–dimensional arithmetic vector space $\mathbb{R}^n$?

- Each system of $n+1$ (or more) vectors is linearly dependent.
- If we get a system of n (or less) vectors, then:
  - construct a matrix consisting of these vectors, and
  - find its rank.
- If the rank equals to the number of vectors, then the system is independent. If the rank is strictly smaller, then the system is dependent.
- Moreover, the system forms a basis if it consists of n linearly independent vectors.

Example

Determine if the vectors $(1, 1, -1)$, $(1, 3, 1)$, $(1, 0, -2)$ from $\mathbb{R}^3$ are linearly dependent or independent.

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$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ -1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Example

Determine if the vectors $(1,1,-1)$, $(1,3,1)$, $(1,0,-2)$ from $\mathbb{R}^3$ are linearly dependent or independent.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ -1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank is 2 and the system is dependent. It cannot form a basis.

Example

Determine if the vectors $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 0, 0, 1)$, $(1, 1, 1, 1)$ from $\mathbb{R}^4$ are linearly dependent or independent.

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$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \cdots$$

Example

$$\cdots \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The rank is 4, the system is linearly independent and forms a basis.

Example

Determine if the vectors $(1, 1, 0, 1)$, $(1, 1, -1, 0)$, $(0, 0, 1, -1)$ from $\mathbb{R}^4$ are linearly dependent or independent.

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Determine if the vectors $(1, 1, 0, 1)$, $(1, 1, -1, 0)$, $(0, 0, 1, -1)$ from $\mathbb{R}^4$ are linearly dependent or independent.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

The rank is $r = 3$, the system is linearly independent (but it does not form a basis).