UMB 551I Linear algebra

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We follow the Czech version of the course (UMB 551 Lineární algebra) by Jan Eisner. The slides in Czech are available on fix.prf.jcu.cz/~eisner/lock/UMB-551/.

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Obsah

We will discuss real or complex matrices, i.e. elements of matrices are real or complex numbers.

Definition

A *matrix of type* $m \times n$ is a rectangular schema A with m rows and n columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where $i = 1, \dots, m$ and $j = 1, \dots, n$. We also use the notation $A = (a_{ij})$.

In fact, the indices describes the position of the element in the schema. More precisely, a_{ij} is the element of the matrix A, which is on the i^{th} row and on j^{th} column.

Example

Consider

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & -2 \end{pmatrix}.$$

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$$a_{12} = 2.$$

- The matrix of type $1 \times n$ of the form $(a_{i1}, a_{i2}, \ldots, a_{in})$ is called the i^{th} *row of the matrix*.
- The matrix of type $m \times 1$ of the form

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

is called the j^{th} *column of the matrix*.

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The first column ($j = 1$) is

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

- If $m = n$, then we speak about a *square matrix*. The number $m = n$ is called the *order of the matrix*.
- The *main diagonal* is $(a_{11}, a_{22}, \ldots, a_{nn})$.

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Example

The matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

is square matrix of order 3 and its main diagonal is $(1, 4, 9)$.

- An *upper triangular matrix* is the matrix A such that everything below the diagonal is zero.
- Analogously, a *lower triangular matrix* is the matrix A such that everything above the diagonal is zero.
- The *zero matrix* is the matrix (of arbitrary type)

$$O = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

- The *identity matrix* or the *unit matrix* is the (always square) matrix

$$E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

- Two matrices are equal if and only if they are of the same type, and the corresponding elements are identical.

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \sin^2 x + \cos^2 x \end{pmatrix} = \begin{pmatrix} \sin^2 x + \cos^2 x & 0 \\ 0 & 1 \end{pmatrix}$$

for each $x \in \mathbb{R}$.

- A *symmetric matrix* is a matrix $A = (a_{ij})$ of order n such that $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$.
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Example

Consider

$$C = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 6 \\ 5 & 6 & 7 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & -6 \\ -1 & 6 & 0 \end{pmatrix}$$

Which of them is symmetric and which of them is antisymmetric?

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C is symmetric, D is antisymmetric

Scalar Multiplication

- Scalar multiplication is defined for arbitrary matrix.
- The product is obtained from multiplying each entry of the matrix by the scalar.

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Example

$$5 \cdot \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 25 \\ 15 & 20 & 30 \end{pmatrix}$$

Matrix Addition

- Matrix addition is defined for matrices of the same type.
- The sum is obtained by adding the corresponding entries of the matrices.

Definition

Let $A = (a_{ij})$, $B = (b_{ij})$ be matrices of the same type. Then

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Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 6 & 3 \end{pmatrix}$$

Matrices of different types cannot be added.

Example

The expression

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 0 \\ 4 & -1 & -2 \end{pmatrix}$$

makes no sense.

Subtracting

- Matrix subtraction is defined for matrices of the same type.
- The difference is obtained by subtraction the corresponding entries of the matrices.

Definition

Let $A = (a_{ij})$, $B = (b_{ij})$ be matrices of the same type. Then

$$A - B = A + (-1)B = (a_{ij} - b_{ij}).$$

Clearly, matrices of different types cannot be subtracted.

Transposition

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Definition

Let A be a matrix of type $m \times n$. The transpose A^T is the matrix of type $n \times m$ whose columns are the rows of A in the same order.

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Example

Consider $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$. Then $A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$.

Theorem

Let A be a square matrix. Then:

- *A is symmetric, if $A = A^T$,*
- *A is anti–symmetric, if $A = -A^T$.*

Moreover, each square matrix A can be written as a sum of symmetric matrix A_s and anti–symmetric matrix A_{as}, where

$$A_s = \frac{1}{2}\left(A + A^T\right), \quad A_{as} = \frac{1}{2}\left(A - A^T\right).$$

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Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

1. Matrices
2. (Simple) Matrix operation
3. Properties of (Simple) Operations
4. Matrix Multiplication

Let A, B, C are matrices and c, d scalars. Suppose the expressions on both sides are defined. There are the following statements:

Theorem

- $A + B = B + A$ *(commutativity)*
- $A + (B + C) = (A + B) + C$ *(associativity)*
- $A + O = O + A = A$
- $A + (-A) = O$

Theorem

- $(c + d)A = cA + dA$
- $c(A + B) = cA + cB$
- $c(dA) = (cd)A$
- $1A = A$
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Theorem

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(cA)^T = cA^T$

Example

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ -2 & 3 \end{pmatrix}$. Find $A^T - B + 2A$.

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We have

$$\begin{aligned} A^T - B + 2A &= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 7 \\ 10 & 9 \end{pmatrix}. \end{aligned}$$

Definition

Let $A = (a_{ij})$ be a matrix of type $m \times n$ and $B = (b_{k\ell})$ be a matrix of type $n \times p$. Put

$$c_{i\ell} := \sum_{k=1}^{n} a_{ik} b_{k\ell}$$

for $i = 1, \ldots, m$ and $\ell = 1, \ldots, p$.
The matrix $C = (c_{i\ell})$ of type $m \times p$ is called a product of the matrix A with the matrix B, tj. $C = AB$.

- The matrix multiplication is only possible if the number of columns of the first matrix equals the number of rows of the second matrix.
- The element of the product of two matrix on the position (i, j) is the dot product of the i^{th} row of the first matrix and the j^{th} column of the second matrix.
- Shortly said: i^{th} row times j^{th} column.

Example

Compute AB and BA, where $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Observation

Since A is of type 2×2 and B is of type 2×3, then

- *AB exists and is of type 2×3,*
- *BA does not exist.*

Example

$$
\begin{aligned}
AB &= \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \\
&= \begin{pmatrix} 0 \cdot 4 + 1 \cdot 7 & 0 \cdot 5 + 1 \cdot 8 & 0 \cdot 6 + 1 \cdot 9 \\ 2 \cdot 4 + 3 \cdot 7 & 2 \cdot 5 + 3 \cdot 8 & 2 \cdot 6 + 3 \cdot 9 \end{pmatrix} \\
&= \begin{pmatrix} 7 & 8 & 9 \\ 29 & 34 & 39 \end{pmatrix}
\end{aligned}
$$

Example

Compute AB and BA, where $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$.

Observation

We can always multiply square matrices of fixed order r, and the result is again of order r. In our situation: $r = 2$. Thus both products exist and are of order 2.

Example

$$AB = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 12 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ 4 & 5 \end{pmatrix}$$

Observation

$AB \neq BA$, i.e. the multiplication is not commutative.

Example

Compute AB and BA, where $A = \begin{pmatrix} -1 & 1 & 2 \end{pmatrix}$ (of type 1×3),

$B = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ (of type 3×1).

Observation

- *AB is of type 1×1, i.e. a scalar*
- *BA is of type 3×3*

Example

$$AB = \begin{pmatrix} -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = -1 \cdot 2 + 1 \cdot 0 + 2 \cdot 3 = \begin{pmatrix} 4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 4 \\ 0 & 0 & 0 \\ -3 & 3 & 6 \end{pmatrix}$$

In general, $AB \neq BA$! In fact, the multiplication is really strange...

Example

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Let A, B, C are matrices and c, d scalars. Suppose the expressions on both sides are defined. There is the following statement:

Theorem

- $A(BC) = (AB)C$ *(associtativity)*
- $(A + B)C = AC + BC$ and $C(A + B) = CA + CB$ *(distributivity)*
- $AE = EA = A$
- $AO = OA = O$
- $(dA)B = A(dB) = d(AB)$
- $(AB)^T = B^T A^T$

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Definition

In particular, if A is square, then we write $A^2 = AA$, $A^3 = AAA$, and so on, where we always use the matrix multiplication.