UMB 551I Linear algebra

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We follow the Czech version (UMB 551 Lineární algebra) by Jan Eisner available on fix.prf.jcu.cz/~eisner/lock/UMB-551/

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Obsah

1. Inverse of a matrix
2. Algorithm
3. Adjugate matrices
4. Solving Linear Systems
5. Matrix equations

Motivation:

- In $\mathbb{R}$, each non–zero real number a has its inverse $a^{-1} = \frac{1}{a}$ for the multiplication such that $a^{-1} \cdot a = 1$.
- Consider square matrices of a fixed order. Then we can multiply these matrices as well, and the role of 1 has the identity matrix, i.e. $EA = AE = A$.
- There is a natural question, whether there is also some 'inverse' for multiplication of matrices. In other words, for a matrix A, is there a matrix A^{-1} such that $A^{-1}A = E$?

Definition

Let A be a square matrix of order n. Then the matrix A^{-1} satisfying $A^{-1}A = E$ and $AA^{-1} = E$ is called an *inverse of the matrix A*.

- Clearly, A^{-1} is a square matrix of order n.
- Inverse of a matrix does not have to exist.
- If there is an inverse A^{-1} of a matrix A, then A is called invertible.

Theorem

Let A be a square matrix of order n. Then there exists inverse A^{-1} of the matrix A if and only if A is regular (i.e. if and only is $|A| \neq 0$ which occurs if and only if $r(A) = n$).

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Let A, B be regular square matrices of order n.

- $\left(A^{-1}\right)^{-1} = A$,
- $|A^{-1}| = \frac{1}{|A|}$,
- $\left(A^{-1}\right)^{T} = \left(A^{T}\right)^{-1}$,
- $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

Algorithm for finding the inverse of a matrix:

- Write the matrix A and the identity matrix E next to each other. Thus we get a matrix $(A|E)$ of type $n \times 2n$.
- Find the row echelon form of the matrix $(A|E)$ using elementary row transformations.
- Use the 'backwards' elimination (i.e. elimination from the bottom and right) to transform the left matrix into the identity matrix.
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We have

$$(A|E) \sim \cdots \sim \left(E|A^{-1}\right).$$

Clearly, if we get a zero row (in the left matrix) during the computation, then the inverse matrix does not exists.

Example

Find the inverse to the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 6 \end{pmatrix}$.

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Firstly, we find the row echelon form:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -1 & 6 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right).$$

Example

Now, we find the diagonal form of the left matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -4 & 2 & 1 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 & 7 & 4 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \sim$$

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Example

We get

$$A^{-1} = \begin{pmatrix} -17 & 7 & 4 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}.$$

Is the result correct?

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Is the result correct? We compute $A \cdot A^{-1} =$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 6 \end{pmatrix} \begin{pmatrix} -17 & 7 & 4 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition

An *adjugate matrix* of the square matrix A (of order n) is the matrix

$$A^* = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix},$$

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i.e. the transpose of the matrix of algebraic complements.

Theorem

Let A be a regular matrix. Then

$$A^{-1} = \frac{1}{|A|} \cdot A^*.$$

Example

Using the adjugate matrix, find A^{-1} for

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$$A^* = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{pmatrix}.$$

Example

Then we have

$$= \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

and we compute

$$|A| = 1 + 2 + 0 - 1 - 0 - 0 = 2.$$

All together,

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}.$$

Consider the system $Ax = b$, where A is square matrix of order n and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

If A is regular, then the system has exactly one solution, and we can find it as follows:

- Multiply the equality $Ax = b$ by the matrix A^{-1} from the left.
- We get $A^{-1}Ax = A^{-1}b$ and we know $A^{-1}A = E$
- Thus the solution is $x = A^{-1}b$.

Example

Using inverse of the matrix of the system, solve the system

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 2 \\ 2x_1 + 3x_2 - x_3 & = & -3 \\ x_1 - x_2 + 6x_3 & = & 14 \end{array}.$$

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We have $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 6 \end{pmatrix}$, $\quad b = \begin{pmatrix} 2 \\ -3 \\ 14 \end{pmatrix}$ and we know from above that $A^{-1} = \begin{pmatrix} -17 & 7 & 4 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$. Then we get

$$x = A^{-1}b = \begin{pmatrix} -17 & 7 & 4 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 14 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

A *matrix equation* is an equation in which a variable stands for a matrix. There are the following types of situations:

- $A + X = B$, where X unknown matrix and A, B, X are of the same type.
- $AX = B$ or $XA = B$, where X is unknown matrix and AX or XA is defined.
- $AXB = C$, where X is unknown matrix and AXB is defined.

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  If the matrix is not regular, we have to use different methods, e.g. to compute explicitly the product AX or XA to get a system of linear equations such that the unknowns are the elements of X.
- $AXB = C$: It depends on the regularity of matrices A and B. If both are regular, the we find inverses A^{-1} and B^{-1}, and multiply the equation by A^{-1} from the left and by B^{-1} from the right to get $A^{-1}AXBB^{-1} = A^{-1}CB^{-1} = X$.
  We will not solve here equations such that at leat one of the matrices A and B is not regular.

Example

Solve the equation $3A + 2X = C - 2B$, where:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 6 & -3 \end{pmatrix}, C = \begin{pmatrix} 5 & 10 & 21 \\ 4 & 15 & 15 \end{pmatrix}.$$

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We get $2X = C - 2B - 3A$, and thus $X = \frac{1}{2}(C - 2B - 3A)$. We compute

$$X = \frac{1}{2}\left[\begin{pmatrix} 5 & 10 & 21 \\ 4 & 15 & 15 \end{pmatrix} - 2\begin{pmatrix} 3 & 1 & 1 \\ 2 & 6 & -3 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 & 3 \\ -2 & 1 & 3 \end{pmatrix}\right]$$

$$= \begin{pmatrix} -2 & 1 & 5 \\ 3 & 0 & 6 \end{pmatrix}.$$

Example

Solve the equation $AX + 2B = BX - 2C$, where

$$A = \begin{pmatrix} -1 & 2 \\ 1 & -5 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ -2 & 8 \end{pmatrix}.$$

Example

Solve the equation $AX + 2B = BX - 2C$, where

$$A = \begin{pmatrix} -1 & 2 \\ 1 & -5 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ -2 & 8 \end{pmatrix}.$$

We have

$$\begin{aligned} AX + 2B &= BX - 2C \\ AX - BX &= -2B - 2C \\ BX - AX &= 2B + 2C \end{aligned}$$

and we get the equation

$$(B - A)X = 2(B + C).$$

Example

The matrix $B - A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 2 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 6 \end{pmatrix}$ is regular, and there is the inverse $(B - A)^{-1}$. Then we multiply the equation by $(B - A)^{-1}$ and we get

$$\begin{aligned} (B - A)^{-1}(B - A)X &= (B - A)^{-1}\, 2(B + C) \\ X &= 2(B - A)^{-1}(B + C). \end{aligned}$$

We compute

$$(B - A)^{-1} = \frac{1}{12}\begin{pmatrix} 6 & 0 \\ -2 & 2 \end{pmatrix}, \quad B + C = \begin{pmatrix} 1 & 3 \\ 1 & 9 \end{pmatrix}$$

and we get

$$X = 2\frac{1}{12}\begin{pmatrix} 6 & 0 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}.$$