UMB 551I Linear algebra

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We follow the Czech version (UMB 551 Lineární algebra) by Jan Eisner available on fix.prf.jcu.cz/~eisner/lock/UMB-551/

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Obsah

Definition

- The system of equations of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

  is called a *system of m linear equations of n variables* $x_1, \dots, x_n$. Here a_{ij}, b_i are reals.
- The *solution* of the system is n–tuple $(t_1, \dots, t_n)$ of real numbers such that all equations are satisfied when we substitute $x_i := t_i$ for $i = 1, \dots, n$.

- a_{ij} ... coefficients of the system
- b_i ... constant terms or absolute values
- $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$ ... matrix of the system
- $\overline{A} = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$ ... extended matrix of the system
- $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ ... vector of absolute values
- $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ... solution vector

Observation: We can write the system using matrices as

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$$Ax = b.$$

Make the explicit computation!

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Definition

- The system is called *consistent* if it has at least one solution. Otherwise, it is called *inconsistent*.
- Two systems are called *equivalent* if they have the same solution set.

Let us point out that ‘to solve the system’ means to find all of its solutions, or to show that it has no solution.
One of the following three possibilities always applies:

- The system has no solution.
- The system has exactly one solution.
- The system has infinite amount of solutions.

Theorem

Each elementary row transformation applied on the extended matrix $(A|b)$ of the system $Ax = b$ leads to the extended matrix of a system, which is equivalent to the original system.

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In other words, the following transformations (applied on the corresponding extended matrix) do not change the solution of the system:

- switching of two equations,
- multiplication of a given equation by a non–zero number,
- adding an equation to another equation.

Gaussian Elimination Method

1. Transform the extended matrix $(A|b)$ of the system $Ax = b$ into the row echelon form.
2. Solve the system corresponding to this matrix in the row echelon form starting with the last (non–trivial) equation and going upwards.
3. Then one of the following three possibilities applies in each step:

(1) In the last non–zero row, there is exactly one non–zero number which is contained in the last column (i.e. in the column corresponding to the absolute values). Then the system has no solution.

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Indeed, the row in question corresponds to the equation

$$0x_1 + 0x_2 + \cdots + 0x_n = \ddagger \neq 0,$$

and the extended matrix is of the form

$$\left(\begin{array}{ccccc|c} * & * & \dots & \dots & * & * \\ 0 & * & \dots & \dots & * & * \\ \vdots & & & & & \\ 0 & 0 & \dots & \dots & * & * \\ 0 & 0 & \dots & \dots & 0 & \ddagger \\ 0 & 0 & \dots & \dots & 0 & 0 \end{array}\right).$$

(2) The actual equation (corresponding to the row of the extended matrix) contains exactly one unknown (with non–zero coefficient), i.e. the remaining ones are already known. Then we simply compute this unknown explicitly. The extended matrix then is

$$\left(\begin{array}{ccccccc|c} . & \dots & . & . & . & \dots & . & . \\ . & \dots & . & . & . & \dots & . & . \\ 0 & \dots & 0 & \clubsuit & \ddagger & \dots & \ddagger & \ddagger \\ . & \dots & . & . & . & \dots & . & . \\ . & \dots & . & . & . & \dots & . & . \end{array}\right),$$

where we know all elements $\ddagger$, and we find $\clubsuit$.

(3) The actual equation (corresponding to the row of the extended matrix) contains more than one unknowns. Then we say that the first unknown (with non–zero coefficient) is **the unknown**, and that the remaining unknowns are **parameters.** The solution than depends on these parameters. The extended matrix then is

$$\left(\begin{array}{cccccccc|c} . & \ldots & . & . & \ldots & . & \ldots & . & . \\ . & \ldots & . & . & \ldots & . & \ldots & . & . \\ 0 & \ldots & 0 & \clubsuit & \diamondsuit & \ddagger & \ldots & \ddagger & \ddagger \\ . & \ldots & . & . & \ldots & . & \ldots & . & . \\ . & \ldots & . & . & \ldots & . & \ldots & . & . \end{array}\right),$$

where we know $\ddagger$, we consider $\diamondsuit$ as parameters, and we find $\clubsuit \neq 0$.

Example

Solve the following system using GEM:

$$
\begin{array}{rcrcrcrcrcr}
 & & 2x_2 & + & 2x_3 & + & 2x_4 & - & 4x_5 & = & 5 \\
x_1 & + & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & 3 \\
-x_1 & - & x_2 & - & x_3 & + & x_4 & + & 2x_5 & = & 0 \\
-2x_1 & + & 3x_2 & + & 3x_3 & & & - & 6x_5 & = & 2
\end{array}
$$

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Solve the following system using GEM:

$$\begin{array}{rcrcrcrcrcr} & & 2x_2 & + & 2x_3 & + & 2x_4 & - & 4x_5 & = & 5 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & 3 \\ -x_1 & - & x_2 & - & x_3 & + & x_4 & + & 2x_5 & = & 0 \\ -2x_1 & + & 3x_2 & + & 3x_3 & & & - & 6x_5 & = & 2 \end{array}$$

We write the extended matrix of the system and find the row echelon form.

Example

$$\begin{pmatrix} 0 & 2 & 2 & 2 & -4 & 5 \\ 1 & 1 & 1 & 1 & -2 & 3 \\ -1 & -1 & -1 & 1 & 2 & 0 \\ -2 & 3 & 3 & 0 & -6 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & -2 & 3 \\ 0 & 2 & 2 & 2 & -4 & 5 \\ -1 & -1 & -1 & 1 & 2 & 0 \\ -2 & 3 & 3 & 0 & -6 & 2 \end{pmatrix} \sim$$

(switch 1st and 2nd row)

Example

$$\begin{pmatrix} 0 & 2 & 2 & 2 & -4 & 5 \\ 1 & 1 & 1 & 1 & -2 & 3 \\ -1 & -1 & -1 & 1 & 2 & 0 \\ -2 & 3 & 3 & 0 & -6 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & -2 & 3 \\ 0 & 2 & 2 & 2 & -4 & 5 \\ -1 & -1 & -1 & 1 & 2 & 0 \\ -2 & 3 & 3 & 0 & -6 & 2 \end{pmatrix} \sim$$

(switch 1st and 2nd row)

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & -2 & 3 \\ 0 & 2 & 2 & 2 & -4 & 5 \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 5 & 5 & 2 & -10 & 8 \end{pmatrix} \sim$$

(3rd row$+$1st row, 4th row$+2 * $1st row)

Example

$$\sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & 1 & 1 & -2 & \frac{5}{2} \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 5 & 5 & 2 & -10 & 8 \end{array}\right) \sim$$

$(1/2 * \text{2nd row})$

Example

$$\sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & 1 & 1 & -2 & \frac{5}{2} \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 5 & 5 & 2 & -10 & 8 \end{array}\right) \sim$$

$(1/2 * 2\text{nd row})$

$$\sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & 1 & 1 & -2 & \frac{5}{2} \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & -3 & 0 & -\frac{9}{2} \end{array}\right) \sim$$

$(4\text{th row} + (-5) * 2\text{rd row})$

Example

$$\sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & 1 & 1 & -2 & \frac{5}{2} \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$(3 * 3\text{rd row} + 2 * 4\text{th row})$

Example

$$\sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & 1 & 1 & -2 & \frac{5}{2} \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$(3 * 3\text{rd row} + 2 * 4\text{th row})$

We solve the system

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & 3 \\ & & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & \frac{5}{2} \\ & & & & & & 2x_4 & & & = & 3 \end{array}$$

Example

We have

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & 3 \\ & & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & \frac{5}{2} \\ & & & & & & 2x_4 & & & = & 3 \end{array},$$

and we start with the last equation and we go upwards:

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and we start with the last equation and we go upwards:

- $x_5 = t$,
- $2x_4 = 3 \quad \Rightarrow \quad x_4 = \frac{3}{2}$

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and we start with the last equation and we go upwards:

- $x_5 = t$,
- $2x_4 = 3 \quad \Rightarrow \quad x_4 = \frac{3}{2}$
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$$\begin{array}{ccccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & 3 \\ & & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & \frac{5}{2} \\ & & & & & & 2x_4 & & & = & 3 \end{array},$$

and we start with the last equation and we go upwards:

- $x_5 = t,$
- $2x_4 = 3 \quad \Rightarrow \quad x_4 = \frac{3}{2}$
- $x_3 = s,$
- $x_2 + s + \frac{3}{2} - 2t = \frac{5}{2} \quad \Rightarrow \quad x_2 = 1 - s + 2t,$

Example

We have

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & 3 \\ & & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & \frac{5}{2} \\ & & & & & & 2x_4 & & & = & 3 \end{array},$$

and we start with the last equation and we go upwards:

- $x_5 = t$,
- $2x_4 = 3 \quad \Rightarrow \quad x_4 = \frac{3}{2}$
- $x_3 = s$,
- $x_2 + s + \frac{3}{2} - 2t = \frac{5}{2} \quad \Rightarrow \quad x_2 = 1 - s + 2t$,
- $x_1 + 1 - s + 2t + s + \frac{3}{2} - 2t = 3 \quad \Rightarrow \quad x_1 = \frac{1}{2}$.

Example

The solution set is of the form

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 - s + 2t \\ s \\ \frac{3}{2} \\ t \end{pmatrix}, \quad s, t \in \mathbb{R} \right\} =$$

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R} \right\}$$

Theorem (Frobenius)

The system $Ax = b$ is consistent (i.e. it has a solution) if and only if the rank of the matrix of the system A equals to the rank of the extended matrix of the system $(A|b)$.

- Moreover, it has exactly one solution, if it is consistent and the rank of A equals to the number of unknowns.
- It has infinite number of solutions, if it is consistent and the rank of A is strictly smaller than number of unknowns.

Definition

The system is called *homogeneous*, if the constant terms are zeros, i.e. b is the zero vector. Otherwise, it is called *non–homogeneous*.

Clearly, the solution set of each homogeneous system always contains the zero vector.

Theorem

Assume we have a homogeneous system $Ax = o$, where A is of type $m \times n$ and o is the zero vector. Then:

- *the system has only zero solution if and only if $r(A)$ equals to n,*
- *the system has some non–zero solution if and only if $r(A)$ is strictly smaller than n.*

In particular, it always suffices to discuss the matrix of the system (and not the extended matrix) for homogeneous systems.

Theorem

Consider the system $Ax = b$. Then:

- *the sum of arbitrary solution of the system $Ax = b$ with arbitrary solution of the system $Ax = o$ is the solution of the system $Ax = b$,*
- *the difference of arbitrary two solutions of the system $Ax = b$ is a solution of the system $Ax = o$.*

The system $Ax = o$ is the so–called *associated homogeneous system* to the system $Ax = b$.

Theorem

For any linear system $Ax = b$, there are vectors $v_1, \ldots, v_k$ such that the solution set can be described as

$$\{v + c_1 v_1 + \cdots + c_k v_k : c_1, \ldots, c_k \in \mathbb{R}\},$$

where v is any particular solution, i.e. arbitrary solution of $Ax = b$, and the system has k parameters.

In fact, vectors $v_1, \ldots, v_k$ are suitable solutions of the associated homogeneous system $Ax = o$ to the system $Ax = b$.