UMB 551I Linear algebra

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We follow the Czech version (UMB 551 Lineární algebra) by Jan Eisner available on fix.prf.jcu.cz/~eisner/lock/UMB-551/

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Obsah

1. Eigenvectors and Eigenvalues
2. How to find Eigenvalues and Eigenvectors?
3. Some more observations

Definition

Let A be a square matrix of order n.

- The number λ is called an *eigenvalue of the matrix A*, if there is a non–zero (column) n–dimensional vector v such that

$$Av = \lambda v.$$

  (There is a matrix multiplication on the left, and the scalar multiplication on the right.)
- The vector v is called an *eigenvector of the matrix A* corresponding to the eigenvalue λ.

If λ is an eigenvalue, then we denote $V_\lambda := \{v \in \mathbb{R}^n \ : \ Av = \lambda v\}$ the set of all eigenvectors corresponding to λ (together with the zero vector).

- A non–trivial subset V of an arithmetic vector space $\mathbb{R}^n$ is called a *subspace*, if the following facts hold:
  1. If $u, v \in V$, then $u + v \in V$,
  2. If $u \in V$, then $ku \in V$ for each $k \in \mathbb{R}$.
- In fact, a subspace is a non–trivial subset of vectors from $\mathbb{R}^n$ which is closed under operations of addition and scalar multiplication, i.e. closed under linear combinations.

Example

In $\mathbb{R}^2$, consider the subset

$$V := \{(t, t) \; : \; t \in \mathbb{R}\}.$$

Geometrically, it is (the set of all vectors lying on) the line $y = x$. Is V a subspace?

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Geometrically, it is (the set of all vectors lying on) the line $y = x$.
Is V a subspace?
The set V is non–trivial, because it contains the origin $(0,0)$.
Moreover, for all $(t,t),(s,s) \in V$ and $r \in \mathbb{R}$ we have

1. $(t,t) + (s,s) = (t+s, t+s) \in V$,
2. $r(t,t) = (rt, rt) \in V$,

Thus it is a subspace.

Theorem

Consider a non–trivial subset of vectors. Then the set of all linear combinations of these vectors form a subspace of an arithmetic vector space, the so–called subspace generated by the subset of vectors.

- In fact, all subspaces in $\mathbb{R}^n$ are lines, planes, spaces, etc. through the origin.
- These are simply solutions of homogeneous systems of n variables.

Theorem

For each eigenvalue λ, the set $V_\lambda = \{v \in \mathbb{R}^n : Av = \lambda v\}$ is a subspace of an arithmetic vector space $\mathbb{R}^n$. It is called an eigenspace with an eigenvalue λ.

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We have

- $o \in V_\lambda$, since $Ao = o = \lambda o$ for each λ,
- $A(u+v) = Au + Av = \lambda u + \lambda v = \lambda(u+v)$ for each $u, v \in V_\lambda$,
- $A(ku) = k(Au) = k\lambda u = \lambda(ku)$ for each $u \in V_\lambda$ and $k \in \mathbb{R}$.

For each eigenvalue λ we have

$$V_\lambda = \{v \in \mathbb{R}^n \,:\, (A - \lambda E)v = 0\}.$$

Thus the space V_λ is the solution of the homogeneous system of linear equations of n variables with the matrix of the system $A - \lambda E$.

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Example

Prove that $\lambda = -3$ is the eigenvalue of the matrix

$$A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$$

and find the corresponding eigenspace V_{-3}.

Example

We find the matrix $A - (-3)E = A + 3E =$

$$\begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{pmatrix}$$

and we solve the corresponding homogeneous system with this matrix (we do not write the absolute values, i.e. zeros). Clearly

$$\begin{pmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and thus $\det A = 0$.

Example

The resulting system is of the form $x_1 + x_2 + 2x_3 = 0$, and its solutions are vectors of the form

$$\begin{pmatrix} s \\ t \\ -\frac{1}{2}(s+t) \end{pmatrix}.$$

For example, the vector $(2, 0, -1)^T$ is an eigenvector with the eigenvalue -3. Thus $\lambda = -3$ is the eigenvalue and

$$V_{-3} = \left\{ s \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} ; s, t \in \mathbb{R} \right\}.$$

Definition

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Theorem

The set of all eigenvalues of the matrix A equals to the set of all roots of the characteristic polynomial of the matrix A.

- Polynomial of degree n has n (generally complex) roots (taken with multiplicities).
- If a real polynomial has a complex root, then it has also the complex conjugated root.

- The matrix A of order n has exactly n eigenvalues (taken with multiplicities).

Theorem

The determinant of the matrix is the product of all eigenvalues (taken with multiplicities).

Let us remark that for a matrix A, the sum of elements on the main diagonal is called a *trace*. It holds that trace equals to the sum of all eigenvalues (taken with multiplicities).

Theorem

All eigenvalues of a symmetric matrix are real.

How to find eigenvalues and eigenvectors of a square matrix A?

1. Find the characteristic polynomial of the matrix A and find all its roots.
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Example

Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Example

We find the polynomial $\det(A - \lambda E)$:

$$A - \lambda E = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$$

$$|A - \lambda E| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2$$

We find roots of the polynomial $-\lambda^3 + 3\lambda + 2$, i.e. we solve the equation

$$-\lambda^3 + 3\lambda + 2 = 0.$$

Example

Possible integer roots of

$$-\lambda^3 + 3\lambda + 2 = 0$$

are divisors of the absolute value 2, i.e. the numbers 1, -1, 2 and -2. We see that $\lambda_1 = -1$ is a root.

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We can also use the so–called *Horner's method*.

	-1	0	3	2
1	-1	-1	2	4
-1	-1	1	2	0

In the last row, there are the coefficients of the polynomial quotient of $-\lambda^3 + 3\lambda + 2$ and $\lambda + 1$, i.e. the polynomial $-\lambda^2 + \lambda + 2$.

Example

We have

$$(-\lambda^3 + 3\lambda + 2) = (\lambda + 1)(-\lambda^2 + \lambda + 2),$$

$$\lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$$

and we get roots $\lambda_2 = -1$, $\lambda_3 = 2$. Thus all eigenvalues of the matrix A are

$$\lambda_1 = \lambda_2 = -1,$$

$$\lambda_3 = 2.$$

Example

We find corresponding eigenspaces for the above eigenvalues:
$\lambda = 2$: We solve the homogeneous system with the matrix $A - 2E$ which is

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly, the system has non–trivial solution. We solve the system

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

and the solution is $V_2 = \left\{ \begin{pmatrix} t \\ t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}.$

Example

$\lambda = -1$: We solve the homogeneous system with the matrix $A + E$ which is

$$A + E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the solution is

$$V_{-1} = \left\{ \begin{pmatrix} -s - t \\ s \\ t \end{pmatrix} : t, s \in \mathbb{R} \right\}$$
$$= \left\{ t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : t, s \in \mathbb{R} \right\}.$$

- There always has to be at least one parameter in the system.
- The number of these parameters is called a *geometric multiplicity of the eigenvalue*.
- The algebraic multiplicity of the eigenvalue then is its multiplicity as a root of the characteristic polynomial.
- These multiplicities do not have to coincide. In fact, geometric multiplicity is always smaller or equal to the algebraic multiplicity.

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Eigenvectors corresponding to different eigenvalues are linearly independent.

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Consider a real matrix A with its (possibly complex) eigenvalues and eigenvectors.
If $\lambda = a + bi$ is eigenvalue of A with eigenvector $u = u_1 + iu_2$, then $\overline{\lambda} = a - bi$ is eigenvalue of A with eigenvector $u = u_1 - iu_2$.

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Theorem

If the matrix A satisfies $A^{-1} = \overline{A}^T$ (the so–called unitary matrix), then all of its eigenvalues have its absolute value equal to 1. In particular, its real eigenvalues can be only 1 or -1.