Eigenvectors and Eigenvalues

How to find Eigenvalues and Eigenvectors?

Some more observations

UMB 551I Linear algebra

Lenka Zalabová

We follow the Czech version (UMB 551 Lineární algebra) by Jan Eisner available on fix.prf.jcu.cz/~eisner/lock/UMB-551/

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Definition

Let $A$ be a square matrix of order $n$.

- The number $\lambda$ is called an eigenvalue of the matrix $A$, if there is a non–zero (column) $n$–dimensional vector $v$ such that

$Av = \lambda v$.

(There is a matrix multiplication on the left, and the scalar multiplication on the right.)

- The vector $v$ is called an eigenvector of the matrix $A$ corresponding to the eigenvalue $\lambda$.

If $\lambda$ is an eigenvalue, then we denote $V_\lambda := \{ v \in \mathbb{R}^n : Av = \lambda v \}$ the set of all eigenvectors corresponding to $\lambda$ (together with the zero vector).
A non–trivial subset \( V \) of an arithmetic vector space \( \mathbb{R}^n \) is called a **subspace**, if the following facts hold:

1. If \( u, v \in V \), then \( u + v \in V \),
2. If \( u \in V \), then \( ku \in V \) for each \( k \in \mathbb{R} \).

In fact, a subspace is a non–trivial subset of vectors from \( \mathbb{R}^n \) which is closed under operations of addition and scalar multiplication, i.e. closed under linear combinations.
Example

In $\mathbb{R}^2$, consider the subset

$$V := \{(t, t) : t \in \mathbb{R}\}.$$  

Geometrically, it is (the set of all vectors lying on) the line $y = x$. Is $V$ a subspace?
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Geometrically, it is (the set of all vectors lying on) the line $y = x$. Is $V$ a subspace?

The set $V$ is non–trivial, because it contains the origin $(0, 0)$. Moreover, for all $(t, t), (s, s) \in V$ and $r \in \mathbb{R}$ we have

1. $(t, t) + (s, s) = (t + s, t + s) \in V,$
2. $r(t, t) = (rt, rt) \in V,$

Thus it is a subspace.
**Theorem**

Consider a non–trivial subset of vectors. Then the set of all linear combinations of these vectors form a subspace of an arithmetic vector space, the so–called subspace generated by the subset of vectors.

- In fact, all subspaces in $\mathbb{R}^n$ are lines, planes, spaces, etc. through the origin.
- These are simply solutions of homogeneous systems of $n$ variables.
Theorem

For each eigenvalue $\lambda$, the set $V_\lambda = \{ v \in \mathbb{R}^n : Av = \lambda v \}$ is a subspace of an arithmetic vector space $\mathbb{R}^n$. It is called an eigenspace with an eigenvalue $\lambda$. 
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We have

- $o \in V_\lambda$, since $Ao = o = \lambda o$ for each $\lambda$,
- $A(u + v) = Au + Av = \lambda u + \lambda v = \lambda(u + v)$ for each $u, v \in V_\lambda$,
- $A(ku) = k(Au) = k\lambda u = \lambda(ku)$ for each $u \in V_\lambda$ and $k \in \mathbb{R}$.
For each eigenvalue \( \lambda \) we have

\[ V_\lambda = \{ v \in \mathbb{R}^n : (A - \lambda E)v = 0 \}. \]

Thus the space \( V_\lambda \) is the solution of the homogeneous system of linear equations of \( n \) variables with the matrix of the system \( A - \lambda E \).
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Thus the space $V_\lambda$ is the solution of the homogeneous system of linear equations of $n$ variables with the matrix of the system $A - \lambda E$.

**Example**

Prove that $\lambda = -3$ is the eigenvalue of the matrix

$$A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$$

and find the corresponding eigenspace $V_{-3}$. 
Example

We find the matrix \( A - (-3)E = A + 3E = \)

\[
\begin{pmatrix}
5 & 8 & 16 \\
4 & 1 & 8 \\
-4 & -4 & -11
\end{pmatrix} + 3 \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
8 & 8 & 16 \\
4 & 4 & 8 \\
-4 & -4 & -8
\end{pmatrix}
\]

and we solve the corresponding homogeneous system with this matrix (we do not write the absolute values, i.e. zeros). Clearly

\[
\begin{pmatrix}
8 & 8 & 16 \\
4 & 4 & 8 \\
-4 & -4 & -8
\end{pmatrix} \sim \begin{pmatrix}
1 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

and thus \( \det A = 0. \)
Example

The resulting system is of the form $x_1 + x_2 + 2x_3 = 0$, and its solutions are vectors of the form

$$\begin{pmatrix} s \\ t \\ -\frac{1}{2}(s + t) \end{pmatrix}.$$  

For example, the vector $(2, 0, -1)^T$ is an eigenvector with the eigenvalue $-3$. Thus $\lambda = -3$ is the eigenvalue and

$$V_{-3} = \left\{ s \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} ; s, t \in \mathbb{R} \right\}.$$
Definition

A *characteristic polynomial* of the square matrix $A$ of order $n$ is defined as the polynomial

$$\det(A - \lambda E)$$

(of degree $n$) with the variable $\lambda$. 
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**Theorem**

*The set of all eigenvalues of the matrix $A$ equals to the set of all roots of the characteristic polynomial of the matrix $A$.*

- Polynomial of degree $n$ has $n$ (generally complex) roots (taken with multiplicities).
- If a real polynomial has a complex root, then it has also the complex conjugated root.
The matrix $A$ of order $n$ has exactly $n$ eigenvalues (taken with multiplicities).

**Theorem**

*The determinant of the matrix is the product of all eigenvalues (taken with multiplicities).*

Let us remark that for a matrix $A$, the sum of elements on the main diagonal is called a *trace*. It holds that trace equals to the sum of all eigenvalues (taken with multiplicities).

**Theorem**

*All eigenvalues of a symmetric matrix are real.*
How to find eigenvalues and eigenvectors of a square matrix $A$?

1. Find the characteristic polynomial of the matrix $A$ and find all its roots.

2. For each eigenvalue $\lambda$, solve the homogeneous system $(A - \lambda E)x = 0$. 

Example: Find all eigenvalues and eigenvectors of the matrix

$$
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
$$

How to find eigenvalues and eigenvectors of a square matrix $A$?

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Example

Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$
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Example

We find the polynomial \( \det(A - \lambda E) \):

\[
A - \lambda E = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix} - \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{pmatrix} = \begin{pmatrix}
-\lambda & 1 & 1 \\
1 & -\lambda & 1 \\
1 & 1 & -\lambda
\end{pmatrix}
\]

\[
|A - \lambda E| = \begin{vmatrix}
-\lambda & 1 & 1 \\
1 & -\lambda & 1 \\
1 & 1 & -\lambda
\end{vmatrix} = -\lambda^3 + 3\lambda + 2
\]

We find roots of the polynomial \( -\lambda^3 + 3\lambda + 2 \), i.e. we solve the equation

\[-\lambda^3 + 3\lambda + 2 = 0.\]
Example

Possible integer roots of

$$-\lambda^3 + 3\lambda + 2 = 0$$

are divisors of the absolute value 2, i.e. the numbers 1, −1, 2 and −2. We see that $\lambda_1 = -1$ is a root.
Example

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are divisors of the absolute value 2, i.e. the numbers 1, –1, 2 and –2. We see that \(\lambda_1 = -1\) is a root.

We can also use the so–called *Horner’s method*.

\[
\begin{array}{cccc}
-1 & 0 & 3 & 2 \\
1 & -1 & -1 & 2 & 4 \\
-1 & -1 & 1 & 2 & 0 \\
\end{array}
\]

In the last row, there are the coefficients of the polynomial quotient of \(-\lambda^3 + 3\lambda + 2\) and \(\lambda + 1\), i.e. the polynomial \(-\lambda^2 + \lambda + 2\).
Example

We have

\[ (-\lambda^3 + 3\lambda + 2) = (\lambda + 1)(-\lambda^2 + \lambda + 2), \]

\[ \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) \]

and we get roots \( \lambda_2 = -1, \lambda_3 = 2 \). Thus all eigenvalues of the matrix \( A \) are

\[ \lambda_1 = \lambda_2 = -1, \]

\[ \lambda_3 = 2. \]
Example

We find corresponding eigenspaces for the above eigenvalues: 
\( \lambda = 2 \): We solve the homogeneous system with the matrix 
\( A - 2E \) which is

\[
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 1 & -2 \\
0 & 3 & -3 \\
0 & 3 & -3
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 1 & -2 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix}
\]

Clearly, the system has non–trivial solution. We solve the system

\[
\begin{align*}
x_1 + x_2 - 2x_3 &= 0 \\
x_2 - x_3 &= 0
\end{align*}
\]

and the solution is

\[
V_2 = \left\{ \begin{pmatrix} t \\ t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}.
\]
Example

\( \lambda = -1 \): We solve the homogeneous system with the matrix \( A + E \) which is

\[
A + E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and the solution is

\[
V_{-1} = \left\{ \begin{pmatrix} -s - t \\ s \\ t \end{pmatrix} : t, s \in \mathbb{R} \right\}
\]

\[
= \left\{ t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : t, s \in \mathbb{R} \right\}.
\]
There always has to be at least one parameter in the system.

The number of these parameters is called a geometric multiplicity of the eigenvalue.

The algebraic multiplicity of the eigenvalue then is its multiplicity as a root of the characteristic polynomial.

These multiplicities do not have to coincide. In fact, geometric multiplicity is always smaller or equal to the algebraic multiplicity.
Theorem

Eigenvectors corresponding to different eigenvalues are linearly independent.
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*Eigenvectors corresponding to different eigenvalues are linearly independent.*

Theorem

*Consider a real matrix $A$ with its (possibly complex) eigenvalues and eigenvectors. If $\lambda = a + bi$ is eigenvalue of $A$ with eigenvector $u = u_1 + iu_2$, then $\bar{\lambda} = a - bi$ is eigenvalue of $A$ with eigenvector $u = u_1 - iu_2$.***
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Theorem

Consider a real matrix $A$ with its (possibly complex) eigenvalues and eigenvectors.
If $\lambda = a + bi$ is eigenvalue of $A$ with eigenvector $u = u_1 + iu_2$, then $\bar{\lambda} = a - bi$ is eigenvalue of $A$ with eigenvector $u = u_1 - iu_2$.

Theorem

If the matrix $A$ satisfies $A^{-1} = \bar{A}^T$ (the so-called unitary matrix), then all of its eigenvalues have its absolute value equal to 1. In particular, its real eigenvalues can be only 1 or −1.