UMB 551I Linear algebra

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We follow the Czech version (UMB 551 Lineární algebra) by Jan Eisner available on fix.prf.jcu.cz/~eisner/lock/UMB-551/

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Obsah

1. Permutations
2. Determinants
3. How to compute determinants?
   - Elementary Transformations
   - Subdeterminants and the Laplace rule
4. Cramer rule

Consider the set $X = \{1, 2, \ldots, n\}$. The bijective map σ from the set X to itself is called a *permutation* of the X.
It is convenient to write a permutation σ into the following table:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(n-1) & \sigma(n) \end{pmatrix}$$

- The first line is the usual linear ordering, while the second row is the rearranging.
- In fact, we can view a permutation as rearranging of the n-tuple $(1, 2, 3, \ldots, n)$.
- The number of such rearranging is exactly $n!$ and we denote the set of all of them by Σ_n.

- The pair $i, j \in X = \{1, 2, \ldots, n\}$ determines an *inversion* in the permutation σ, if $i < j$ and $\sigma(i) > \sigma(j)$.
- A *parity (signature)* $\operatorname{sgn}(\sigma)$ of the permutation σ is the parity of the number of inversions for σ, i.e.

$$\operatorname{sgn}(\sigma) = (-1)^{\text{number of inversions}}.$$

Then we have the following terminology:

- odd permutation ... $\operatorname{sgn}(\sigma) = -1$,
- even permutation ... $\operatorname{sgn}(\sigma) = 1$.

Example

Find all permutations of the set $X = \{1, 2, 3\}$ and find their signatures.

Example

Find all permutations of the set $X = \{1, 2, 3\}$ and find their signatures. Since we have the set $X = \{1, 2, 3\}$, the number of permutations is $3! = 6$. Then we have:

- $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, no inversion, $\operatorname{sgn}(\sigma) = 1$, even,
- $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, one inversion $\left| \begin{matrix} 2 < 3 \\ 3 > 2 \end{matrix} \right|$, $\operatorname{sgn}(\sigma) = -1$, odd,
- $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, one inversion $\left| \begin{matrix} 1 < 2 \\ 2 > 1 \end{matrix} \right|$, $\operatorname{sgn}(\sigma) = -1$, odd,

Example

- $\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, three inversions
  $\left| \begin{matrix} 1<2 & 1<3 & 2<3 \\ 3>2 & 3>1 & 2>1 \end{matrix} \right|$, $\operatorname{sgn}(\sigma) = -1$, odd,
- $\sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, two inversions $\left| \begin{matrix} 1<2 & 1<3 \\ 3>1 & 3>2 \end{matrix} \right|$,
  $\operatorname{sgn}(\sigma) = 1$, even,
- $\sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, two inversions $\left| \begin{matrix} 1<3 & 2<3 \\ 2>1 & 3>1 \end{matrix} \right|$,
  $\operatorname{sgn}(\sigma) = 1$, even.

Definition

Let $A = (a_{ij})$ be a square matrix of order n. A *determinant* of the matrix A is the number denoted by $\det A$ or $|A|$ and defined by

$$|A| = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

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- We sum here over all permutations of the set $\{1, 2, \ldots, n\}$.
- We choose elements of the matrix such that:
  1. from the 1st row we take the $\sigma(1)$st element,
  2. from the 2nd row we take the $\sigma(2)$nd element,
  3. etc.

  and we multiply the elements.
- In fact, we have to take one element from each row and from each column.
- The fact that we are 'working' over all permutations means that we 'consider' all such choices.
- Then we add and subtract the products according to the signature of the concrete choice (i.e. concrete permutation).

Example

Find the determinant of the general matrix of order 3.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Example

We have 6 permutations, i.e. 6 summands.

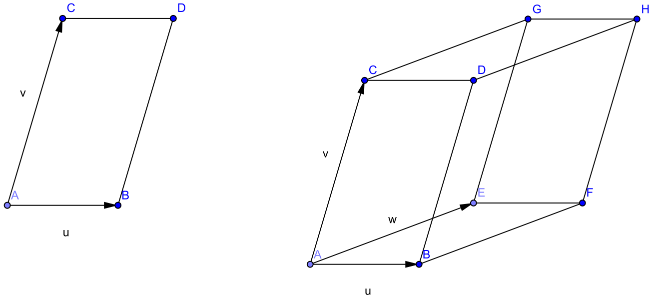
$$\begin{pmatrix} \clubsuit & \cdot & \cdot \\ \cdot & \clubsuit & \cdot \\ \cdot & \cdot & \clubsuit \end{pmatrix} - \begin{pmatrix} \clubsuit & \cdot & \cdot \\ \cdot & \cdot & \clubsuit \\ \cdot & \clubsuit & \cdot \end{pmatrix} - \begin{pmatrix} \cdot & \clubsuit & \cdot \\ \clubsuit & \cdot & \cdot \\ \cdot & \cdot & \clubsuit \end{pmatrix} -$$

$$\begin{pmatrix} \cdot & \cdot & \clubsuit \\ \cdot & \clubsuit & \cdot \\ \clubsuit & \cdot & \cdot \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & \clubsuit \\ \clubsuit & \cdot & \cdot \\ \cdot & \clubsuit & \cdot \end{pmatrix} + \begin{pmatrix} \cdot & \clubsuit & \cdot \\ \cdot & \cdot & \clubsuit \\ \clubsuit & \cdot & \cdot \end{pmatrix}$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} -$$

$$a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} .$$

Clearly, this is not the way to compute determinants. There are more effective methods to find the determinant of a matrix.



The determinant is the area of the parallelogram, the volume of the parallelepiped, and so on for the higher–dimensional analogs.

We find the determinant of the matrix of order 2 using the so–called *cross rule*.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$\searrow$ add
$\swarrow$ subtract

We find the determinant of the matrix of order 3 using the so–called *rule of Sarrus*.

$$\begin{array}{|ccc|} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \quad \begin{aligned} = \quad & a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ & - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21} \end{aligned}$$

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}$$

$\searrow$ add
$\swarrow$ subtract

In fact, this is the computation from the definition.

Example

Find the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

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$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \begin{matrix} \\ \\ \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix} \begin{aligned} &= 1\cdot 5\cdot 9 + 4\cdot 8\cdot 3 + 7\cdot 2\cdot 6 \\ &-3\cdot 5\cdot 7 - 6\cdot 8\cdot 1 - 9\cdot 2\cdot 4 \\ &= 45 + 96 + 84 - 105 - 48 - 72 = 0. \end{aligned}$$

Such methods does not work in the case of matrices of higher order!

Theorem

Let A be a matrix of order n. Then the following statements holds:

1. $|A| = |A^T|$
2. *If there is a zero row in the matrix A, then $|A| = 0$.*
3. *If B differ by A by switching of arbitrary two rows, then $|B| = -|A|$.*
4. *If B differ by A by multiplying of an arbitrary row by a non–zero number a, then $|B| = a|A|$.*
5. *The determinant $|A|$ does not change, if we add to a row an arbitrary linear combination of the remaining rows.*

The point (1) says that (2)–(5) holds for columns, too.

Theorem

The determinant of the matrix A in the echelon form equals to the product of all elements on the main diagonal, i.e.

$$|A| = a_{11} a_{22} \dots a_{nn}.$$

Clearly, tho only non–zero permutation is exactly the diagonal. The remaining permutations have to meet the upper and the lower (zero) triangle.

The previous observations give an effective method of computation of determinants.

- We transform the matrix into the echelon form.
- We have to control the elementary transformations that change the value of the determinant.
- Then we find the product of elements on the diagonal of the matrix in the echelon form (together with possible correctiones given by transformations changing the value of the determinant).

Example

Find the determinant

$$\begin{vmatrix} 3 & -2 & 1 & -2 \\ -3 & -5 & 2 & 0 \\ 2 & 1 & -2 & -4 \\ -1 & 0 & 3 & 1 \end{vmatrix}$$

Example

$$\begin{vmatrix} 3 & -2 & 1 & -2 \\ -3 & -5 & 2 & 0 \\ 2 & 1 & -2 & -4 \\ -1 & 0 & 3 & 1 \end{vmatrix} \text{switch 1st and 4th row}$$

$$= - \begin{vmatrix} -1 & 0 & 3 & 1 \\ -3 & -5 & 2 & 0 \\ 2 & 1 & -2 & -4 \\ 3 & -2 & 1 & -2 \end{vmatrix} \quad \begin{matrix} \text{(1st row) } -3^* \text{ (1st row)} \\ \text{(3rd row) } +2^* \text{ (1st row)} \\ \text{(4th row) } +3^* \text{ (1st row)} \end{matrix}$$

$$= - \begin{vmatrix} -1 & 0 & 3 & 1 \\ 0 & -5 & -7 & -3 \\ 0 & 1 & 4 & -2 \\ 0 & -2 & 10 & 1 \end{vmatrix} \text{switch 2nd a 3rd row}$$

Example

$$= \begin{vmatrix} -1 & 0 & 3 & 1 \\ 0 & 1 & 4 & -2 \\ 0 & -5 & -7 & -3 \\ 0 & -2 & 10 & 1 \end{vmatrix} \quad \begin{matrix} \text{(3rd row)} + 5^* \text{ (2nd row)} \\ \text{(4th row)} + 2^* \text{ (2nd row)} \end{matrix}$$

$$= \begin{vmatrix} -1 & 0 & 3 & 1 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 13 & -13 \\ 0 & 0 & 18 & -3 \end{vmatrix}$$

$$= 13 \begin{vmatrix} -1 & 0 & 3 & 1 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 18 & -3 \end{vmatrix} \quad \text{(4th row)} - 18^* \text{ (3rd row)}$$

$$= 13 \begin{vmatrix} -1 & 0 & 3 & 1 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 15 \end{vmatrix} = 13 \cdot (-1 \cdot 1 \cdot 1 \cdot 15) = 13 \cdot (-15) = -195.$$

- Let $A = (a_{ij})$ be a matrix of type $m \times n$ and let $1 \leq i_1 < \cdots < i_k \leq m$, $1 \leq j_1 < \cdots < j_\ell \leq n$ are arbitrary fixed integers. Then the matrix

$$M = \begin{pmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \dots & a_{i_1 j_\ell} \\ \vdots & & & \\ a_{i_k j_1} & a_{i_k j_2} & \dots & a_{i_k j_\ell} \end{pmatrix}$$

  of type $k \times \ell$ is called a *submatrix* of the matrix A determined by the rows $i_1, \dots, i_k$ and the columns $j_1, \dots, j_\ell$.
- The remaining $(m-k)$ rows and $(n-\ell)$ columns determine the matrix M^* of type $(m-k) \times (n-\ell)$, which is called *complementary submatrix of M in A*.
- If $k = \ell$, then there is the determinant $|M|$, which is called a *minor* of order k of the matrix A.
- If simultaneously $m = n$ and $k = \ell$, then M^* is a square matrix, too, and its determinant $|M^*|$ is called a *complementary minor* to $|M|$ in A.

Example

For the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ -1 & -2 & -3 & -4 & -5 \\ -6 & -7 & -8 & -9 & -10 \\ 0 & 11 & 12 & 13 & 14 \end{pmatrix},$$

write the submatrix M given by rows $2, 3, 5$ and columns $1, 4, 5$. Write the complementary submatrix M^* and find the minors $|M|$ a $|M^*|$ (if they exist).

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For the matrix

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write the submatrix M given by rows $2, 3, 5$ and columns $1, 4, 5$. Write the complementary submatrix M^* and find the minors $|M|$ a $|M^*|$ (if they exist).

The matrix M is of type 3×3 and has the form

$$M = \begin{pmatrix} a_{21} & a_{24} & a_{25} \\ a_{31} & a_{34} & a_{35} \\ a_{51} & a_{54} & a_{55} \end{pmatrix} = \begin{pmatrix} 6 & 9 & 10 \\ -1 & -4 & -5 \\ 0 & 13 & 14 \end{pmatrix}.$$

Example

Then the matrix M^* is of type 2×2 and has the form

$$M = \begin{pmatrix} a_{12} & a_{13} \\ a_{42} & a_{43} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -7 & -8 \end{pmatrix}.$$

Example

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The minors are $|M| = 50$ and $|M^*| = 5$ (and $|A| = 0$).

- The number $(-1)^{i_1+\cdots+i_k+j_1+\cdots+j_\ell}|M^*|$ is called an *algebraic complement to the minor* $|M|$.
- If $m = n$ and $k = \ell = 1$ (i.e. we choose one row and one column), we speak about the *algebraic complement of the element* a_{ij} of the matrix A and we denote it by A_{ij}.

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Example

For the matrix A from above, write A_{34}.

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- If $m = n$ and $k = \ell = 1$ (i.e. we choose one row and one column), we speak about the *algebraic complement of the element* a_{ij} of the matrix A and we denote it by A_{ij}.

Example

For the matrix A from above, write A_{34}.
We simply remove 3rd row and 4th column and we get

$$A_{34} = (-1)^{3+4}\begin{vmatrix} 1 & 2 & 3 & 5 \\ 6 & 7 & 8 & 10 \\ -6 & -7 & -8 & -10 \\ 0 & 11 & 12 & 14 \end{vmatrix}$$

(and this equals to zero).

There is the following method for finding the determinant using submatrices and minors. (It is useful for matrices containing zeros.)

Theorem (Laplace expansion)

For the matrix A of order n and for arbitrary row or column, respectively, we have

$$|A| = \sum_{j=1}^{n} a_{ij} A_{ij},$$

(Laplace expansion for ith row)

$$|A| = \sum_{i=1}^{n} a_{ij} A_{ij}$$

(Laplace expansion for jth column).

Example

Using the Laplace expansion, find the determinant

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

Example

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} \text{(expansion for the 1st row)} =$$

$$2 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 1 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$+0 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} + 0 \cdot (-1)^{1+4} \cdot \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

Example

$$= 2 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 1 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

(expansion for the 1st row, expansion for the 1st column)

$$= 2 \cdot (-1)^{1+1} \cdot \left(2 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 1 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}\right)$$

$$+1 \cdot (-1)^{1+2} \cdot 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 2 \cdot (2 \cdot (4-1) - (2-0)) - (4-1)$$

$$= 2 \cdot (2 \cdot 3 - 2) - 3$$

$$= 2 \cdot 4 - 3 = 8 - 3 = 5.$$

There is the following more general statement:

Theorem (Laplace)

Let A be a square matrix of order n and fix arbitrary k its rows. Then $|A|$ equals to the sum of all $\binom{n}{k}$ products of minors of order k choosed from the fixed rows, together with their algebraic complements.

(This is not so usefull for computations).

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Let A be a square matrix of order n and fix arbitrary k its rows. Then $|A|$ equals to the sum of all $\binom{n}{k}$ products of minors of order k choosed from the fixed rows, together with their algebraic complements.

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Corollary (Cauchy)

Let A, B be square matrices of order n. Then

$$|AB| = |A| \cdot |B|.$$

There is not agalogous statement for ading of matrices! In general, $|A + B| \neq |A| + |B|$.

Definition

The square matrix is called *regular*, if $|A| \neq 0$. If $|A| = 0$, then the matrix A is called *singular*.

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Theorem (Rule of Cramer)

Consider a system of n linear equations of n unknowns such that the matrix of the system is regular. Then the system has exactly one solution $(x_1, \ldots, x_n)$ *of the form*

$$x_j = \frac{|A_j|}{|A|}, \quad j = 1, 2, \ldots, n,$$

where A_j *is the matrix formed by replacing the* j*th column of* A *by the column vector of absolute values.*

Example

Solve the following system using the Cramer rule:

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 + x_2 - x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 1 \end{aligned}$$

Example

Solve the following system using the Cramer rule:

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 + x_2 - x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 1 \end{aligned}$$

We compute the determinant of the matrix of the system

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & -1 \end{pmatrix},$$

$|A| = -1 + 1 + (-2) - 2 + 1 + 1 = -2,$

Example

$$A_1 = \begin{pmatrix} 6 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}, \quad |A_1| = -6 + 0 - 1 - 1 + 6 - 0 = -2,$$

$$x_1 = \frac{-2}{-2} = 1,$$

$$A_2 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix}, \quad |A_2| = 0 + 1 - 12 - 0 + 1 + 6 = -4,$$

$$x_2 = \frac{-4}{-2} = 2,$$

$$A_3 = \begin{pmatrix} 1 & 1 & 6 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad |A_3| = 1 + 6 + 0 - 12 - 1 - 0 = -6,$$

$$x_3 = \frac{-6}{-2} = 3,$$

The solution is the vector $(1, 2, 3)$.