3

One-Parameter Bifurcations of Equilibria in Continuous-Time Dynamical Systems

In this chapter we formulate conditions defining the simplest bifurcations of equilibria in n-dimensional continuous-time systems: the fold and the Hopf bifurcations. Then we study these bifurcations in the lowest possible dimensions: the fold bifurcation for scalar systems and the Hopf bifurcaof equilibria in n-dimensional continuous-time systems: the fold and the Hopf bifurcations. Then we study these bifurcations in the lowest possible dimensions: the fold bifurcation for scalar systems and the Hopf bifurcation for planar systems. Chapter 5 shows how to "lift" these results to n-dimensional situations.

3.1 Simplest bifurcation conditions

Consider a continuous-time system depending on a parameter

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^1,$$

where f is smooth with respect to both x and α. Let $x = x_0$ be a hyperbolic equilibrium in the system for $\alpha = \alpha_0$. As we have seen in Chapter 2, under a small parameter variation the equilibrium moves slightly but remains hyperbolic. Therefore, we can vary the parameter further and monitor the equilibrium. It is clear that there are, generically, only two ways in which the hyperbolicity condition can be violated. Either a simple real eigenvalue approaches zero and we have $\lambda_1 = 0$ (see Figure 3.1(a)), or a pair of simple complex eigenvalues reaches the imaginary axis and we have $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$ (see Figure 3.1(b)) for some value of the parameter. It is obvious (and can be rigorously formalized) that we need more parameters to allocate extra eigenvalues on the imaginary axis. Notice that this might

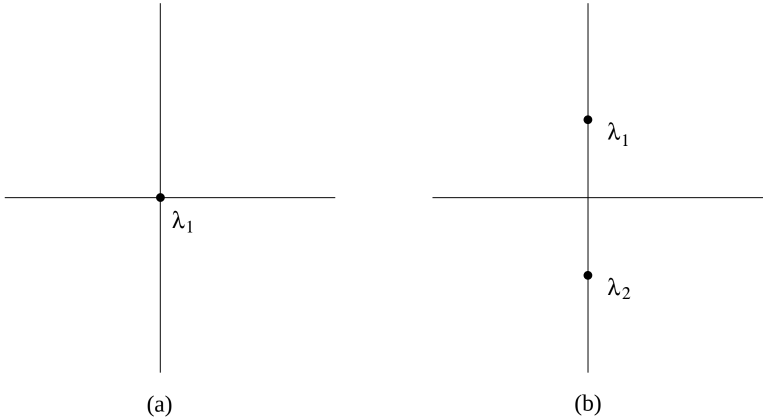

FIGURE 3.1. Codim 1 critical cases.

not be true if the system has some special properties, such as a symmetry (see Chapter 7).

The rest of the chapter will essentially be devoted to the proof that a nonhyperbolic equilibrium satisfying one of the above conditions is structurally *unstable* and to the analysis of the corresponding bifurcations of the local phase portrait under variation of the parameter. We have already seen several examples of these bifurcations in Chapter 2. Let us finish this section with the following two definitions.

Definition 3.1 *The bifurcation associated with the appearance of* $\lambda_1 = 0$ *is called a* fold (*or* tangent) *bifurcation.*

Remark:

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Remark:

This bifurcation has a lot of other names, including *limit point, saddle-node bifurcation*, and *turning point.* ♢

Definition 3.2 *The bifurcation corresponding to the presence of* $\lambda_{1,2} = \pm i\omega_0$*,* $\omega_0 > 0$*, is called a* Hopf (*or* Andronov-Hopf) *bifurcation.*

Notice that the tangent bifurcation is possible if $n \geq 1$, but for the Hopf bifurcation we need $n \geq 2$.

3.2 The normal form of the fold bifurcation

Consider the following one-dimensional dynamical system depending on one parameter:

$$\dot{x} = \alpha + x^2 \equiv f(x, \alpha). \tag{3.1}$$

At $\alpha = 0$ this system has a nonhyperbolic equilibrium $x_0 = 0$ with $\lambda = f_x(0,0) = 0$. The behavior of the system for all the other values of α is also clear (see Figure 3.2). For $\alpha < 0$ there are two equilibria in the system: $x_{1,2}(\alpha) = \pm\sqrt{-\alpha}$, the left one of which is stable, while the right one is unstable. For $\alpha > 0$ there are no equilibria in the system. While α crosses zero from negative to positive values, the two equilibria (stable

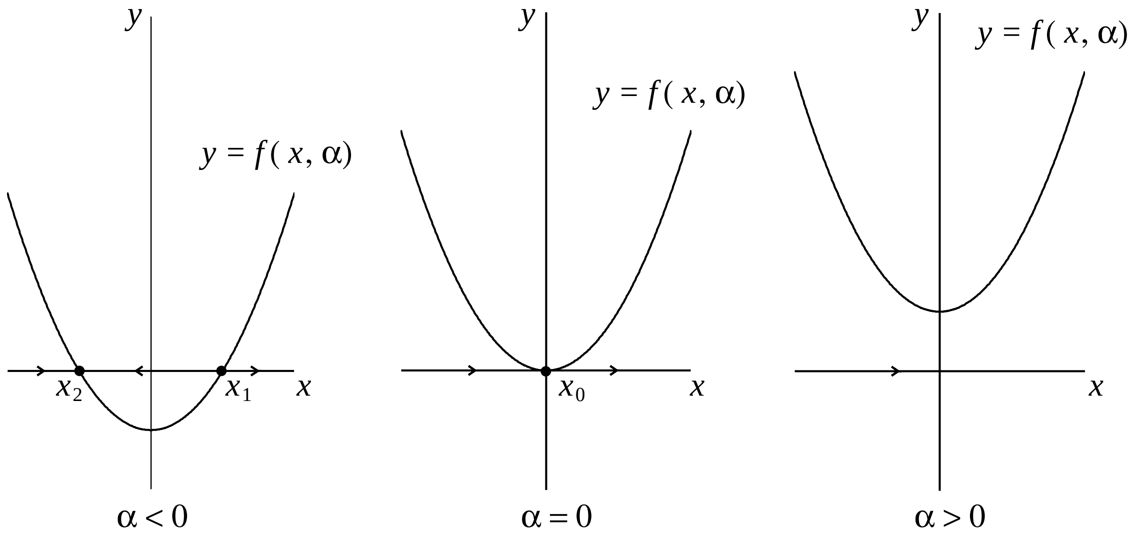

FIGURE 3.2. Fold bifurcation.

and unstable) "collide," forming at $\alpha = 0$ an equilibrium with $\lambda = 0$, and disappear. This is a fold bifurcation. The term "collision" is appropriate, since the speed of approach $\left(\frac{d}{d\alpha}x_{1,2}(\alpha)\right)$ of the equilibria tends to infinity as $\alpha \to 0$.

There is another way of presenting this bifurcation: plotting a bifurcation diagram in the direct product of the phase and parameter spaces (simply, the (x, α)-plane). The equation

$$f(x, \alpha) = 0$$

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$$f(x, \alpha) = 0$$

defines an *equilibrium manifold*, which is simply the parabola $\alpha = -x^2$ (see Figure 3.3). This presentation displays the bifurcation picture at once. Fixing some α, we can easily determine the number of equilibria in the

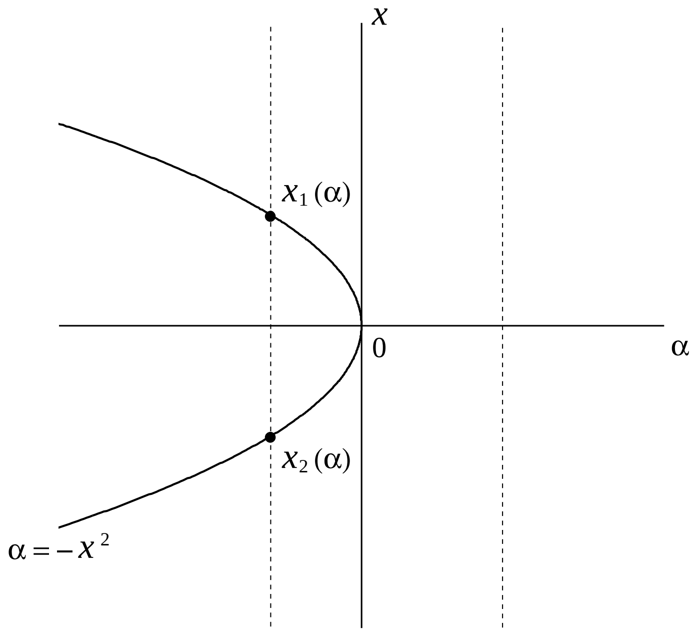

FIGURE 3.3. Fold bifurcation in the phase-parameter space.

system for this parameter value. The projection of the equilibrium manifold into the parameter axis has a *singularity* of the fold type at $(x, \alpha) = (0, 0)$.

Remark:
The system $\dot{x} = \alpha - x^2$ can be considered in the same way. The analysis reveals two equilibria appearing for $\alpha > 0$. $\diamondsuit$

Now add to system (3.1) higher-order terms that can depend smoothly on the parameter. It happens that these terms do not change qualitatively the behavior of the system near the origin $x = 0$ for parameter values close to $\alpha = 0$. Actually, the following lemma holds:

Lemma 3.1 *The system*

$$\dot{x} = \alpha + x^2 + O(x^3)$$

is locally topologically equivalent near the origin to the system

$$\dot{x} = \alpha + x^2.$$

Proof:
The proof goes through two steps. It is based on the fact that for scalar systems a homeomorphism mapping equilibria into equilibria will also map their connecting orbits.

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Step 1 (*Analysis of equilibria*). Introduce a scalar variable y and write the first system as

$$\dot{y} = F(y, \alpha) = \alpha + y^2 + \psi(y, \alpha), \tag{3.2}$$

where $\psi = O(y^3)$ is a smooth functions of (y, α) near $(0, 0)$. Consider the equilibrium manifold of (3.2) near the origin $(0, 0)$ of the (y, α)-plane:

$$M = \{(y, \alpha) : F(y, \alpha) = \alpha + y^2 + \psi(y, \alpha) = 0\}.$$

The curve M passes through the origin $(F(0, 0) = 0)$. By the Implicit Function Theorem (since $F_\alpha(0, 0) = 1$), it can be locally parametrized by y:

$$M = \{(y, \alpha) : \alpha = g(y)\},$$

where g is smooth and defined for small $|y|$. Moreover,

$$g(y) = -y^2 + O(y^3)$$

(check!). Thus, for any sufficiently small $\alpha < 0$, there are two equilibria of (3.2) near the origin in (3.2), $y_1(\alpha)$ and $y_2(\alpha)$, which are close to the equilibria of (3.1), i.e., $x_1(\alpha) = +\sqrt{-\alpha}$ and $x_2(\alpha) = -\sqrt{-\alpha}$, for the same parameter value (see Figure 3.4).

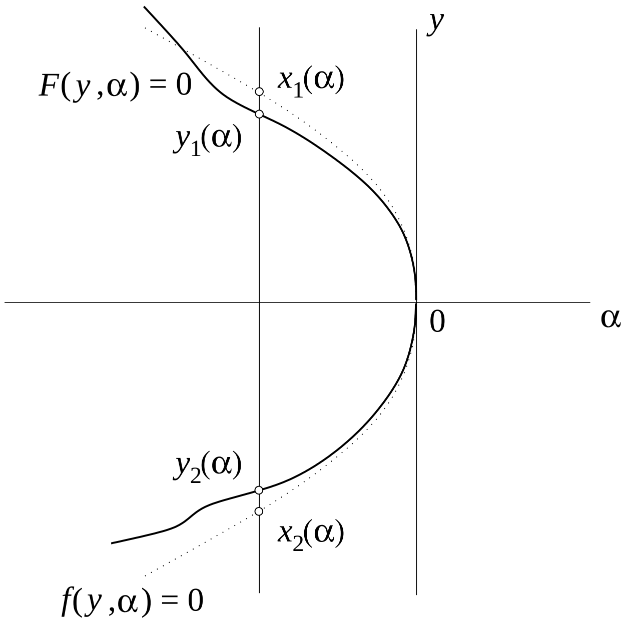

FIGURE 3.4. Fold bifurcation for the perturbed system.

Step 2 (*Homeomorphism construction*). For small $|\alpha|$, construct a parameter-dependent map $y = h_\alpha(x)$ as following. For $\alpha \geq 0$ take the identity map

$$h_\alpha(x) = x.$$

For $\alpha < 0$ take a linear transformation

$$h_\alpha(x) = a(\alpha) + b(\alpha)x,$$

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$$h_\alpha(x) = a(\alpha) + b(\alpha)x,$$

where the coefficients a, b are uniquely determined by the conditions

$$h_\alpha(x_j(\alpha)) = y_j(\alpha), \quad j = 1, 2,$$

(find them!). The constructed map $h_\alpha : \mathbb{R}^1 \to \mathbb{R}^1$ is a homeomorphism mapping orbits of (3.1) near the origin into the corresponding orbits of (3.2), preserving the direction of time. Chapter 2 identified this property as the *local topological equivalence* of parameter-dependent systems.

Although it is not required in the book for the homeomorphism h_α to depend continuously on α (see Remark after Definition 2.14), this property holds here, since h_α tends to the identity map as negative $\alpha \to 0$. □

3.3 Generic fold bifurcation

We shall show that system (3.1) (with a possible sign change of the x^2-term) is a topological normal form of a generic one-dimensional system having a fold bifurcation. In Chapter 5 we will also see that in some strong sense it describes the fold bifurcation in a generic n-dimensional system.

Suppose the system

$$\dot{x} = f(x, \alpha), \qquad x \in \mathbb{R}^1, \ \alpha \in \mathbb{R}^1, \tag{3.3}$$

with a smooth f has at $\alpha = 0$ the equilibrium $x = 0$ with $\lambda = f_x(0, 0) = 0$. Expand $f(x, \alpha)$ as a Taylor series with respect to x at $x = 0$:

$$f(x, \alpha) = f_0(\alpha) + f_1(\alpha)x + f_2(\alpha)x^2 + O(x^3).$$

Two conditions are satisfied: $f_0(0) = f(0, 0) = 0$ (*equilibrium condition*) and $f_1(0) = f_x(0, 0) = 0$ (*fold bifurcation condition*).

The main idea of the following simple calculations is this: By smooth invertible changes of the coordinate and the parameter, transform system (3.3) into the form (3.1) up to and including the second-order terms. Then, Lemma 3.1 can be applied, thus making it possible to drop the higher-order terms. While proceeding, we will see that some extra *nondegeneracy* and *transversality conditions* must be imposed to make these transformations possible. These conditions will actually specify which one-parameter system having a fold bifurcation can be considered as *generic*. This idea works for all local bifurcation problems. We will proceed in exactly this way in analyzing the Hopf bifurcation later in this chapter.

Step 1 (*Shift of the coordinate*). Perform a linear coordinate shift by introducing a new variable ξ:

$$\xi = x + \delta, \tag{3.4}$$

where $\delta = \delta(\alpha)$ is an a priori unknown function that will be defined later. The inverse coordinate transformation is

$$x = \xi - \delta$$

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Substituting (3.4) into (3.3) yields

$$\dot{\xi} = \dot{x} = f_0(\alpha) + f_1(\alpha)(\xi - \delta) + f_2(\alpha)(\xi - \delta)^2 + \cdots .$$

Therefore,

$$\begin{aligned}
\dot{\xi} \quad = \quad & \left[f_0(\alpha) - f_1(\alpha)\delta + f_2(\alpha)\delta^2 + O(\delta^3)\right] \\
& + \left[f_1(\alpha) - 2f_2(\alpha)\delta + O(\delta^2)\right]\xi \\
& + \left[f_2(\alpha) + O(\delta)\right]\xi^2 \\
& + O(\xi^3).
\end{aligned}$$

Assume that

$$f_2(0) = \frac{1}{2} f_{xx}(0,0) \neq 0. \tag{A.1}$$

Then there is a smooth function $\delta(\alpha)$ that annihilates the linear term in the above equation for all sufficiently small $|\alpha|$. This can be justified with the Implicit Function Theorem. Indeed, the condition for the linear term to vanish can be written as

$$F(\alpha, \delta) \equiv f_1(\alpha) - 2f_2(\alpha)\delta + \delta^2\psi(\alpha, \delta) = 0$$

with some smooth function ψ. We have

$$F(0,0) = 0, \quad \left.\frac{\partial F}{\partial \delta}\right|_{(0,0)} = -2f_2(0) \neq 0, \quad \left.\frac{\partial F}{\partial \alpha}\right|_{(0,0)} = f_1'(0),$$

which implies (local) existence and uniqueness of a smooth function $\delta = \delta(\alpha)$ such that $\delta(0) = 0$ and $F(\alpha, \delta(\alpha)) \equiv 0$. It also follows that

$$\delta(\alpha) = \frac{f_1'(0)}{2f_2(0)}\alpha + O(\alpha^2).$$

The equation for ξ now contains no linear terms:

$$\dot{\xi} = [f_0'(0)\alpha + O(\alpha^2)] + [f_2(0) + O(\alpha)]\xi^2 + O(\xi^3). \tag{3.5}$$

Step 2 (*Introduce a new parameter*). Consider as a new parameter $\mu = \mu(\alpha)$ the constant (ξ-independent) term of (3.5):

$$\mu = f_0'(0)\alpha + \alpha^2\phi(\alpha),$$

where ϕ is some smooth function. We have:

(a) $\mu(0) = 0$;
(b) $\mu'(0) = f_0'(0) = f_\alpha(0,0)$.

If we assume that

(b) $\mu'(0) = f_0'(0) = f_\alpha(0,0)$.

If we assume that

$$f_\alpha(0,0) \neq 0, \tag{A.2}$$

then the Inverse Function Theorem implies local existence and uniqueness of a smooth inverse function $\alpha = \alpha(\mu)$ with $\alpha(0) = 0$. Therefore, equation (3.5) now reads

$$\dot{\xi} = \mu + a(\mu)\xi^2 + O(\xi^3),$$

where $a(\mu)$ is a smooth function with $a(0) = f_2(0) \neq 0$ due to the first assumption (A.1).

Step 3 (*Final scaling*). Let $\eta = |a(\mu)|\xi$ and $\beta = |a(\mu)|\mu$. Then we get

$$\dot{\eta} = \beta + s\eta^2 + O(\eta^3),$$

where $s = \text{sign } a(0) = \pm 1$.

Therefore, the following theorem is proved.

Theorem 3.1 *Suppose that a one-dimensional system*

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^1, \ \alpha \in \mathbb{R}^1,$$

with smooth f, has at $\alpha = 0$ the equilibrium $x = 0$, and let $\lambda = f_x(0,0) = 0$. Assume that the following conditions are satisfied:

(A.1) $f_{xx}(0,0) \neq 0$;
(A.2) $f_{\alpha}(0,0) \neq 0$.

Then there are invertible coordinate and parameter changes transforming the system into

$$\dot{\eta} = \beta \pm \eta^2 + O(\eta^3). \ \square$$

Using Lemma 3.1, we can eliminate $O(\eta^3)$ terms and finally arrive at the following general result.

Theorem 3.2 (Topological normal form for the fold bifurcation) *Any generic scalar one-parameter system*

$$\dot{x} = f(x, \alpha),$$

having at $\alpha = 0$ *the equilibrium* $x = 0$ *with* $\lambda = f_x(0,0) = 0$, *is locally topologically equivalent near the origin to one of the following normal forms:*

$$\dot{\eta} = \beta \pm \eta^2. \ \square$$

Remark:
The genericity conditions in Theorem 3.2 are the nondegeneracy condition (A.1) and the transversality condition (A.2) from Theorem 3.1. $\diamondsuit$

The genericity conditions in Theorem 3.2 are the nondegeneracy condition (A.1) and the transversality condition (A.2) from Theorem 3.1. ♢

3.4 The normal form of the Hopf bifurcation

Consider the following system of two differential equations depending on one parameter:

$$\begin{cases} \dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2). \end{cases} \tag{3.6}$$

This system has the equilibrium $x_1 = x_2 = 0$ for all α with the Jacobian matrix

$$A = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}$$

having eigenvalues $\lambda_{1,2} = \alpha \pm i$. Introduce the complex variable $z = x_1 + ix_2, \bar{z} = x_1 - ix_2, |z|^2 = z\bar{z} = x_1^2 + x_2^2$. This variable satisfies the differential equation

$$\dot{z} = \dot{x}_1 + i\dot{x}_2 = \alpha(x_1 + ix_2) + i(x_1 + ix_2) - (x_1 + ix_2)(x_1^2 + x_2^2),$$

and we can therefore rewrite system (3.6) in the following *complex* form:

$$\dot{z} = (\alpha + i)z - z|z|^2. \tag{3.7}$$

Finally, using the representation $z = \rho e^{i\varphi}$, we obtain

$$\dot{z} = \dot{\rho} e^{i\varphi} + \rho i \dot{\varphi} e^{i\varphi},$$

or

$$\dot{\rho} e^{i\varphi} + i\rho\dot{\varphi} e^{i\varphi} = \rho e^{i\varphi}(\alpha + i - \rho^2),$$

which gives the *polar* form of system (3.6):

$$\begin{cases} \dot{\rho} &= \rho(\alpha - \rho^2), \\ \dot{\varphi} &= 1. \end{cases} \tag{3.8}$$

Bifurcations of the phase portrait of the system as α passes through zero can easily be analyzed using the polar form, since the equations for ρ and φ in (3.8) are uncoupled. The first equation (which should obviously be considered only for $\rho \geq 0$) has the equilibrium point $\rho = 0$ for all values of α. The equilibrium is linearly stable if $\alpha < 0$; it remains stable at $\alpha = 0$ but *nonlinearly* (so the rate of solution convergence to zero is no longer exponential); for $\alpha > 0$ the equilibrium becomes linearly unstable. Moreover, there is an additional stable equilibrium point $\rho_0(\alpha) = \sqrt{\alpha}$ for $\alpha > 0$. The second equation describes a rotation with constant speed. Thus, by superposition of the motions defined by the two equations of (3.8), we obtain the following bifurcation diagram for the original two-dimensional system (3.6) (see Figure 3.5). The system always has an equilibrium at the origin. This equilibrium is a stable focus for $\alpha < 0$ and an unstable focus for $\alpha > 0$. At the critical parameter value $\alpha = 0$ the equilibrium is nonlinearly stable (see Figure 3.5). The system always has an equilibrium at the origin. This equilibrium is a stable focus for $\alpha < 0$ and an unstable focus for $\alpha > 0$. At the critical parameter value $\alpha = 0$ the equilibrium is nonlinearly stable and topologically equivalent to the focus. Sometimes it is called a *weakly attracting focus*. This equilibrium is surrounded for $\alpha > 0$ by an isolated closed orbit (*limit cycle*) that is unique and stable. The cycle is a circle of radius $\rho_0(\alpha) = \sqrt{\alpha}$. All orbits starting outside or inside the cycle except at the origin tend to the cycle as $t \to +\infty$. This is an Andronov-Hopf bifurcation.

This bifurcation can also be presented in (x, y, α)-space (see Figure 3.6). The appearing α-family of limit cycles forms a *paraboloid* surface.

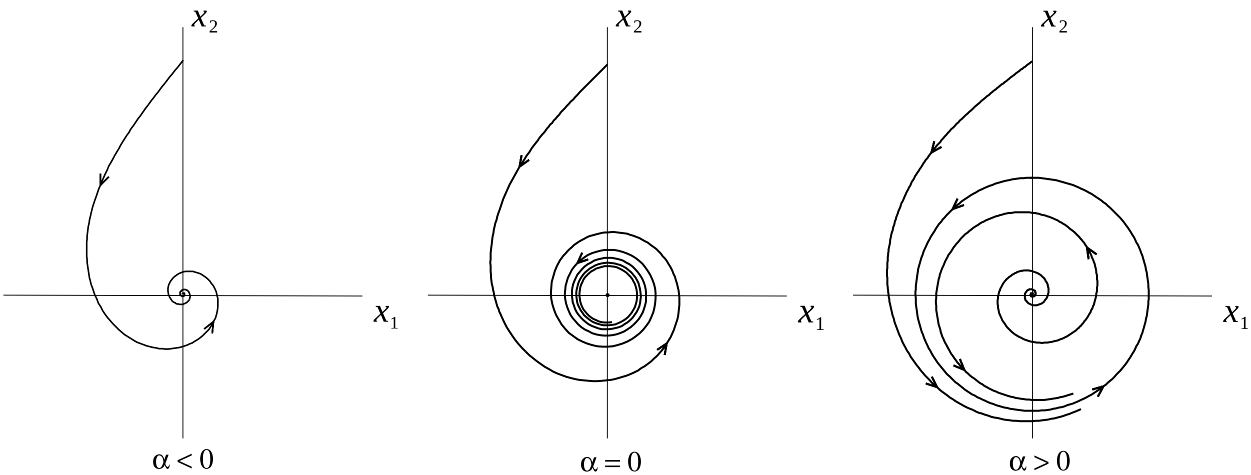

FIGURE 3.5. Supercritical Hopf bifurcation.

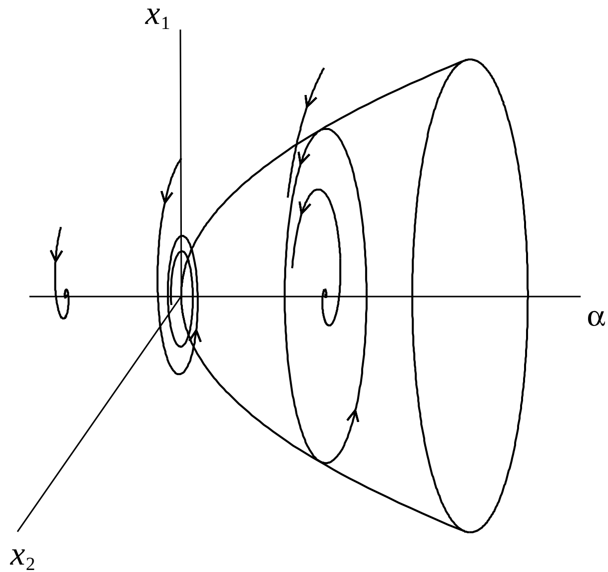

FIGURE 3.6. Supercritical Hopf bifurcation in the phase-parameter space.

A system having nonlinear terms with the opposite sign,

$$\begin{cases} \dot{x}_1 = \alpha x_1 - x_2 + x_1(x_1^2 + x_2^2), \\ \dot{x}_2 = x_1 + \alpha x_2 + x_2(x_1^2 + x_2^2), \end{cases} \tag{3.9}$$

which has the following complex form:

$$\dot{z} = (\alpha + i)z + z|z|^2,$$

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can be analyzed in the same way (see Figures 3.7 and 3.8). The system undergoes the Andronov-Hopf bifurcation at $\alpha = 0$. Contrary to system (3.6), there is an *unstable* limit cycle in (3.9), which disappears when α crosses zero from negative to positive values. The equilibrium at the origin has the same stability for $\alpha \neq 0$ as in system (3.6): It is stable for $\alpha < 0$ and unstable for $\alpha > 0$. Its stability at the critical parameter value is opposite to that in (3.6): It is (nonlinearly) unstable at $\alpha = 0$.

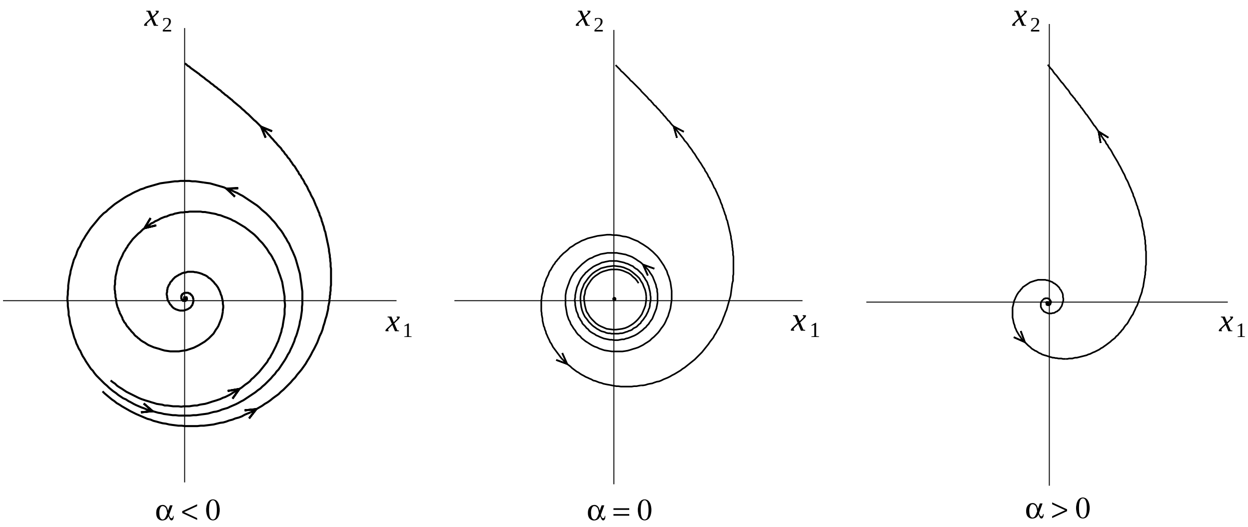


FIGURE 3.7. Subcritical Hopf bifurcation.

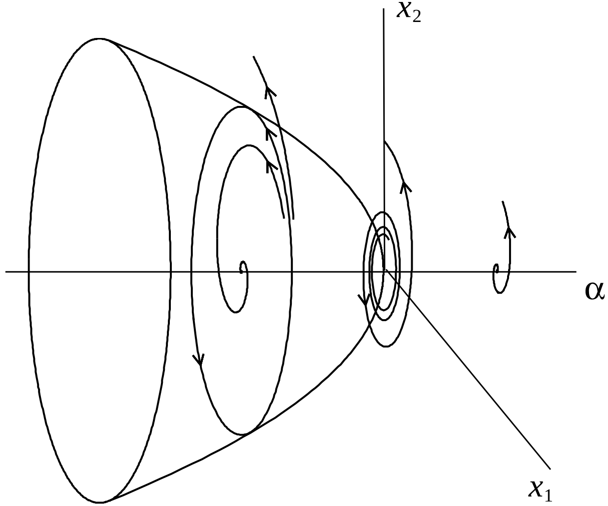


FIGURE 3.8. Subcritical Hopf bifurcation in the phase-parameter space.

Remarks:

(1) We have seen that there are *two* types of Andronov-Hopf bifurcation. The bifurcation in system (3.6) is often called *supercritical* because the cycle exists for positive values of the parameter α ("after" the bifurcation). The bifurcation in system (3.9) is called *subcritical* since the cycle is present "before" the bifurcation. It is clear that this terminology is somehow misleading since "after" and "before" depend on the chosen direction of parameter variation.

(2) In both cases we have a *loss of stability* of the equilibrium at $\alpha = 0$

how misleading since “after” and “before” depend on the chosen direction of parameter variation.

(2) In both cases we have a *loss of stability* of the equilibrium at $\alpha = 0$ under increase of the parameter. In the first case (with “$-$” in front of the cubic terms), the stable equilibrium is replaced by a stable limit cycle of small amplitude. Therefore, the system “remains” in a neigborhood of the equilibrium and we have a *soft* or *noncatastrophic* stability loss. In the second case (with “$+$” in front of the cubic terms), the region of attraction of the equilibrium point is bounded by the unstable cycle, which “shrinks” as the parameter approaches its critical value and disappears. Thus, the system is “pushed out” from a neigborhood of the equilibrium, giving us a *sharp* or *catastrophic* loss of stability. If the system loses stability softly, it is well “controllable”: If we make the parameter negative again, the system returns to the stable equilibrium. On the contrary, if the system loses its stability sharply, resetting to a negative value of the parameter may not return the system back to the stable equilibrium since it may have left its region of attraction. Notice that the type of Andronov-Hopf bifurcation is determined by the stability of the equilibrium at the critical parameter value.

(3) The above interpretation of super- and subcritical Hopf bifurcations should be considered with care. If we consider α as a slow variable and add to system (3.6) the third equation

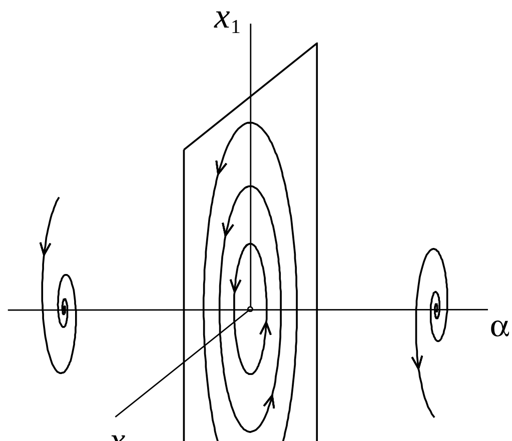
$$\dot{\alpha} = \varepsilon,$$

with ε small but positive, then the resulting *time series* $(x(t), y(t), \alpha(t))$ will demonstrate some degree of "sharpness." If the solution starts at some initial point (x_0, y_0, α_0) with $\alpha_0 < 0$, it then converges to the origin and remains very close to it even if α becomes positive, thus demonstrating no oscillations. Only when α reaches some finite positive value will the solution leave the equilibrium "sharply" and start to oscillate with a relatively large amplitude.

(4) Finally, consider a system without nonlinear terms:

$$\dot{z} = (\alpha + i)z.$$

This system also has a family of periodic orbits of increasing amplitude, but all of them are present at $\alpha = 0$ when the system has a *center* at the origin (see Figure 3.9). It can be said that the limit cycle paraboloid "degenerates"



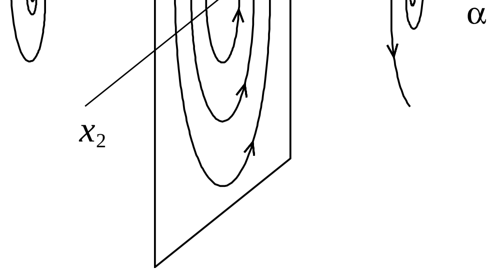


FIGURE 3.9. "Hopf bifurcation" in a linear system.

into the plane $\alpha = 0$ in (x, y, α)-space in this case. This observation makes natural the appearance of small limit cycles in the nonlinear case. ♢

Let us now add some higher-order terms to system (3.6) and write it in the vector form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (x_1^2 + x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + O(\|x\|^4), \quad (3.10)$$

where $x = (x_1, x_2)^T$, $\|x\|^2 = x_1^2 + x_2^2$, and $O(\|x\|^4)$ terms can smoothly depend on α. The following lemma will be proved in Appendix 1 to this chapter.

Lemma 3.2 *System* (3.10) *is locally topologically equivalent near the origin to system* (3.6). □

Therefore, the higher-order terms do not affect the bifurcation behavior of the system.

3.5 Generic Hopf bifurcation

We now shall prove that any generic two-dimensional system undergoing a Hopf bifurcation can be transformed into the form (3.10) with a possible difference in the sign of the cubic terms.

Consider a system

$$\dot{x} = f(x, \alpha), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1,$$

with a smooth function f, which has at $\alpha = 0$ the equilibrium $x = 0$ with eigenvalues $\lambda_{1,2} = \pm i\omega_0, \ \omega_0 > 0$. By the Implicit Function Theorem, the system has a unique equilibrium $x_0(\alpha)$ in some neigborhood of the origin for all sufficiently small $|\alpha|$, since $\lambda = 0$ is not an eigenvalue of the Jacobian matrix. We can perform a coordinate shift, placing this equilibrium at the origin. Therefore, we may assume without loss of generality that $x = 0$ is the equilibrium point of the system for $|\alpha|$ sufficiently small. Thus, the system can be written as

$$\dot{x} = A(\alpha)x + F(x, \alpha), \tag{3.11}$$

where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in x starting with at least quadratic terms, $F = O(\|x\|^2)$. The Jacobian matrix $A(\alpha)$ can be written as

$$A(\alpha) = \begin{pmatrix} a(\alpha) & b(\alpha) \\ c(\alpha) & d(\alpha) \end{pmatrix}$$

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with smooth functions of α as its elements. Its eigenvalues are the roots of the characteristic equation

$$\lambda^2 - \sigma\lambda + \Delta = 0,$$

where $\sigma = \sigma(\alpha) = a(\alpha) + d(\alpha) = \text{tr } A(\alpha)$, and $\Delta = \Delta(\alpha) = a(\alpha)d(\alpha) - b(\alpha)c(\alpha) = \det A(\alpha)$. So,

$$\lambda_{1,2}(\alpha) = \frac{1}{2}\left(\sigma(\alpha) \pm \sqrt{\sigma^2(\alpha) - 4\Delta(\alpha)}\right).$$

The Hopf bifurcation condition implies

$$\sigma(0) = 0, \;\; \Delta(0) = \omega_0^2 > 0.$$

For small $|\alpha|$ we can introduce

$$\mu(\alpha) = \frac{1}{2}\sigma(\alpha), \;\; \omega(\alpha) = \frac{1}{2}\sqrt{4\Delta(\alpha) - \sigma^2(\alpha)}$$

and therefore obtain the following representation for the eigenvalues:

$$\lambda_1(\alpha) = \lambda(\alpha), \;\; \lambda_2(\alpha) = \overline{\lambda(\alpha)},$$

where

$$\lambda(\alpha) = \mu(\alpha) + i\omega(\alpha), \;\; \mu(0) = 0, \;\; \omega(0) = \omega_0 > 0.$$

Lemma 3.3 *By introducing a complex variable z, system (3.11) can be written for sufficiently small $|\alpha|$ as a single equation:*

$$\dot{z} = \lambda(\alpha)z + g(z, \bar{z}, \alpha), \tag{3.12}$$

where $g = O(|z|^2)$ is a smooth function of $(z, \bar{z}, \alpha)$.

Proof:

Let $q(\alpha) \in \mathbb{C}^2$ be an eigenvector of $A(\alpha)$ corresponding to the eigenvalue $\lambda(\alpha)$:

$$A(\alpha)q(\alpha) = \lambda(\alpha)q(\alpha),$$

and let $p(\alpha) \in \mathbb{C}^2$ be an eigenvector of the transposed matrix $A^T(\alpha)$ corresponding to its eigenvalue $\overline{\lambda(\alpha)}$:

$$A^T(\alpha)p(\alpha) = \overline{\lambda(\alpha)}p(\alpha).$$

It is always possible to normalize p *with respect to* q:

$$\langle p(\alpha), q(\alpha)\rangle = 1,$$

where $\langle \cdot, \cdot \rangle$ means the standard scalar product in $\mathbb{C}^2$: $\langle p, q\rangle = \bar{p}_1 q_1 + \bar{p}_2 q_2$. Any vector $x \in \mathbb{R}^2$ can be uniquely represented for any small α as

$$x = zq(\alpha) + \bar{z}\bar{q}(\alpha) \tag{3.13}$$

for some complex z, provided the eigenvectors are specified. Indeed, we have an *explicit* formula to determine z:

for some complex z, provided the eigenvectors are specified. Indeed, we have an *explicit* formula to determine z:

$$z = \langle p(\alpha), x \rangle.$$

To verify this formula (which results from taking the scalar product with p of both sides of (3.13)), we have to prove that $\langle p(\alpha), \bar{q}(\alpha) \rangle = 0$. This is the case, since

$$\langle p, \bar{q} \rangle = \langle p, \frac{1}{\bar{\lambda}} A\bar{q} \rangle = \frac{1}{\bar{\lambda}} \langle A^T p, \bar{q} \rangle = \frac{\lambda}{\bar{\lambda}} \langle p, \bar{q} \rangle$$

and therefore

$$\left(1 - \frac{\lambda}{\bar{\lambda}}\right) \langle p, \bar{q} \rangle = 0.$$

But $\lambda \neq \bar{\lambda}$ because for all sufficiently small $|\alpha|$ we have $\omega(\alpha) > 0$. Thus, the only possibility is $\langle p, \bar{q} \rangle = 0$.

The complex variable z obviously satisfies the equation

$$\dot{z} = \lambda(\alpha) z + \langle p(\alpha), F(zq(\alpha) + \bar{z}\bar{q}(\alpha), \alpha) \rangle,$$

having the required[1] form (3.12) with

$$g(z, \bar{z}, \alpha) = \langle p(\alpha), F(zq(\alpha) + \bar{z}\bar{q}(\alpha), \alpha) \rangle. \ \square$$

---

[1] The vectors $q(\alpha)$ and $p(\alpha)$, corresponding to the simple eigenvalues, can be selected to depend on α as smooth as $A(\alpha)$.

There is no reason to expect g to be an analytic function of z (i.e., $\bar{z}$-independent). Write g as a formal Taylor series in two complex variables (z and $\bar{z}$):

$$g(z, \bar{z}, \alpha) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(\alpha) z^k \bar{z}^l,$$

where

$$g_{kl}(\alpha) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} \langle p(\alpha), F(zq(\alpha) + \bar{z}\bar{q}(\alpha), \alpha) \rangle \bigg|_{z=0},$$

for $k + l \geq 2,\ k, l = 0, 1, \ldots$.

Remarks:

(1) There are several (equivalent) ways to prove Lemma 3.3. The selected one fits well into the framework of Chapter 5, where we will consider the Hopf bifurcation in n-dimensional systems.

(2) Equation (3.13) imposes a linear relation between (x_1, x_2) and the real and imaginary parts of z. Thus, the introduction of z can be viewed as a linear invertible change of variables, $y = T(\alpha)x$, and taking $z = y_1 + iy_2$. As it can be seen from (3.13), the components (y_1, y_2) are the coordinates of x in the *real eigenbasis* of $A(\alpha)$ composed by $\{2 \text{ Re } q, -2 \text{ Im } q\}$. In this basis, the matrix $A(\alpha)$ has its *canonical real* (*Jordan*) *form*:

$$J(\alpha) = T(\alpha)A(\alpha)T^{-1}(\alpha) = \begin{pmatrix} \mu(\alpha) & -\omega(\alpha) \\ \omega(\alpha) & \mu(\alpha) \end{pmatrix}.$$

$$J(\alpha) = T(\alpha)A(\alpha)T^{-1}(\alpha) = \begin{pmatrix} \mu(\alpha) & -\omega(\alpha) \\ \omega(\alpha) & \mu(\alpha) \end{pmatrix}.$$

(3) Suppose that at $\alpha = 0$ the function $F(x, \alpha)$ from (3.11) is represented as

$$F(x, 0) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(\|x\|^4),$$

where $B(x, y)$ and $C(x, y, u)$ are *symmetric multilinear* vector functions of $x, y, u \in \mathbb{R}^2$. In coordinates, we have

$$B_i(x, y) = \sum_{j,k=1}^{2} \left. \frac{\partial^2 F_i(\xi, 0)}{\partial \xi_j \partial \xi_k} \right|_{\xi=0} x_j y_k, \;\; i = 1, 2,$$

and

$$C_i(x, y, u) = \sum_{j,k,l=1}^{2} \left. \frac{\partial^3 F_i(\xi, 0)}{\partial \xi_j \partial \xi_k \partial \xi_l} \right|_{\xi=0} x_j y_k u_l, \;\; i = 1, 2.$$

Then,

$$B(zq + \bar{z}\bar{q}, zq + \bar{z}\bar{q}) = z^2 B(q, q) + 2z\bar{z}B(q, \bar{q}) + \bar{z}^2 B(\bar{q}, \bar{q}),$$

where $q = q(0), p = p(0)$, so the Taylor coefficients $g_{kl},\;\; k + l = 2$, of the quadratic terms in $g(z, \bar{z}, 0)$ can be expressed by the formulas

$$g_{20} = \langle p, B(q, q)\rangle, \;\;\; g_{11} = \langle p, B(q, \bar{q})\rangle, \;\;\; g_{02} = \langle p, B(\bar{q}, \bar{q})\rangle,$$

and similar calculations with C give

$$g_{21} = \langle p, C(q, q, \bar{q}) \rangle.$$

(4) The normalization of q is irrelevant in the following. Indeed, suppose that q is normalized by $\langle q, q \rangle = 1$. A vector $\tilde{q} = \gamma q$ is also the eigenvector for any nonzero $\gamma \in \mathbb{C}^1$ but with the normalization $\langle \tilde{q}, \tilde{q} \rangle = |\gamma|^2$. Taking $\tilde{p} = \frac{1}{\bar{\gamma}} p$ will keep the relative normalization untouched: $\langle \tilde{p}, \tilde{q} \rangle = 1$. It is clear that Taylor coefficients $\tilde{g}_{kl}$ computed using $\tilde{q}, \tilde{p}$ will be *different* from the original g_{kl}. For example, we can check via the multilinear representation that

$$\tilde{g}_{20} = \gamma g_{20},\ \tilde{g}_{11} = \bar{\gamma} g_{11},\ \tilde{g}_{02} = \frac{\bar{\gamma}^2}{\gamma} g_{02},\ \tilde{g}_{21} = |\gamma|^2 g_{21}.$$

However, this change can easily be neutralized by the linear scaling of the variable: $z = \frac{1}{\gamma} w$, which results in the same equation for w as before.

For example, setting $\langle q, q \rangle = \frac{1}{2}$ corresponds to the standard relation $z = \langle p, x \rangle = x_1 + i x_2$ for a system that already has the real canonical form $\dot{x} = J(\alpha) x$, where J is given above. In this case,

$$q = \frac{1}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix},\quad p = \begin{pmatrix} 1 \\ -i \end{pmatrix}. \ \diamondsuit$$

Let us start to make *nonlinear* (complex) coordinate changes that will

Let us start to make *nonlinear* (complex) coordinate changes that will simplify (3.12). First of all, remove all quadratic terms.

Lemma 3.4 *The equation*

$$\dot{z} = \lambda z + \frac{g_{20}}{2} z^2 + g_{11} z\bar{z} + \frac{g_{02}}{2} \bar{z}^2 + O(|z|^3), \tag{3.14}$$

where $\lambda = \lambda(\alpha) = \mu(\alpha) + i\omega(\alpha), \mu(0) = 0, \omega(0) = \omega_0 > 0,$ *and* $g_{ij} = g_{ij}(\alpha)$, *can be transformed by an invertible parameter-dependent change of complex coordinate*

$$z = w + \frac{h_{20}}{2} w^2 + h_{11} w\bar{w} + \frac{h_{02}}{2} \bar{w}^2,$$

for all sufficiently small $|\alpha|$, *into an equation without quadratic terms:*

$$\dot{w} = \lambda w + O(|w|^3).$$

Proof:

The inverse change of variable is given by the expression

$$w = z - \frac{h_{20}}{2} z^2 - h_{11} z\bar{z} - \frac{h_{02}}{2} \bar{z}^2 + O(|z|^3).$$

Therefore,

$$\dot{w} = \dot{z} - h_{20} z\dot{z} - h_{11}(\dot{z}\bar{z} + z\dot{\bar{z}}) - h_{02}\bar{z}\dot{\bar{z}} + \cdots$$

$$= \lambda z + \left(\frac{g_{20}}{2} - \lambda h_{20}\right) z^2 + \left(g_{11} - \lambda h_{11} - \bar{\lambda} h_{11}\right) z\bar{z} + \left(\frac{g_{02}}{2} - \bar{\lambda} h_{02}\right) \bar{z}^2 + \cdots$$

$$= \lambda w + \frac{1}{2}(g_{20} - \lambda h_{20}) w^2 + (g_{11} - \bar{\lambda} h_{11}) w\bar{w} + \frac{1}{2}(g_{02} - (2\bar{\lambda} - \lambda) h_{02}) \bar{w}^2 + O(|w|^3).$$

Thus, by setting

$$h_{20} = \frac{g_{20}}{\lambda}, \ \ h_{11} = \frac{g_{11}}{\bar{\lambda}}, \ \ h_{02} = \frac{g_{02}}{2\bar{\lambda} - \lambda},$$

we "kill" all the quadratic terms in (3.14). These substitutions are correct because the denominators are nonzero for all sufficiently small $|\alpha|$ since $\lambda(0) = i\omega_0$ with $\omega_0 > 0$. □

Remarks:

(1) The resulting coordinate transformation is polynomial with coefficients that are smoothly dependent on α. The inverse transformation has the same property but it is not polynomial. Its form can be obtained by the method of unknown coefficients. In some neighborhood of the origin the transformation is *near-identical* because of its linear part.

(2) Notice that the transformation *changes* the coefficients of the cubic (as well as higher-order) terms of (3.14). ◇

Assuming that we have removed all quadratic terms, let us try to eliminate the cubic terms as well. This is "almost" possible: There is only one

Assuming that we have removed all quadratic terms, let us try to eliminate the cubic terms as well. This is "almost" possible: There is only one "resistant" term, as the following lemma shows.

Lemma 3.5 *The equation*

$$\dot{z} = \lambda z + \frac{g_{30}}{6} z^3 + \frac{g_{21}}{2} z^2 \bar{z} + \frac{g_{12}}{2} z \bar{z}^2 + \frac{g_{03}}{6} \bar{z}^3 + O(|z|^4),$$

where $\lambda = \lambda(\alpha) = \mu(\alpha) + i\omega(\alpha), \mu(0) = 0, \omega(0) = \omega_0 > 0,$ *and* $g_{ij} = g_{ij}(\alpha)$, *can be transformed by an invertible parameter-dependent change of complex coordinate*

$$z = w + \frac{h_{30}}{6} w^3 + \frac{h_{21}}{2} w^2 \bar{w} + \frac{h_{12}}{2} w \bar{w}^2 + \frac{h_{03}}{6} \bar{w}^3,$$

for all sufficiently small $|\alpha|$, *into an equation with only one cubic term:*

$$\dot{w} = \lambda w + c_1 w^2 \bar{w} + O(|w|^4),$$

where $c_1 = c_1(\alpha)$.

Proof:

The inverse transformation is

$$w = z - \frac{h_{30}}{6} z^3 - \frac{h_{21}}{2} z^2 \bar{z} - \frac{h_{12}}{2} z \bar{z}^2 - \frac{h_{03}}{6} \bar{z}^3 + O(|z|^4).$$

Therefore,

$$\begin{aligned}\dot{w} &= \dot{z} - \frac{h_{30}}{2}z^2\dot{z} - \frac{h_{21}}{2}(2z\bar{z}\dot{z} + z^2\dot{\bar{z}}) - \frac{h_{12}}{2}(\dot{z}\bar{z}^2 + 2z\bar{z}\dot{\bar{z}}) - \frac{h_{03}}{2}\bar{z}^2\dot{\bar{z}} + \cdots \\ &= \lambda z + \left(\frac{g_{30}}{6} - \frac{\lambda h_{30}}{2}\right)z^3 + \left(\frac{g_{21}}{2} - \lambda h_{21} - \frac{\bar{\lambda}h_{21}}{2}\right)z^2\bar{z} \\ &\quad + \left(\frac{g_{12}}{2} - \frac{\lambda h_{12}}{2} - \bar{\lambda}h_{12}\right)z\bar{z}^2 + \left(\frac{g_{03}}{6} - \frac{\bar{\lambda}h_{03}}{2}\right)\bar{z}^3 + \cdots \\ &= \lambda w + \frac{1}{6}(g_{30} - 2\lambda h_{30})w^3 + \frac{1}{2}(g_{21} - (\lambda + \bar{\lambda})h_{21})w^2\bar{w} \\ &\quad + \frac{1}{2}(g_{12} - 2\bar{\lambda}h_{12})w\bar{w}^2 + \frac{1}{6}(g_{03} + (\lambda - 3\bar{\lambda})h_{03})\bar{w}^3 + O(|w|^4).\end{aligned}$$

Thus, by setting

$$h_{30} = \frac{g_{30}}{2\lambda},\ h_{12} = \frac{g_{12}}{2\bar{\lambda}},\ h_{03} = \frac{g_{03}}{3\bar{\lambda} - \lambda},$$

we can annihilate all cubic terms in the resulting equation except the $w^2\bar{w}$ -term, which we have to treat separately. The substitutions are valid since all the involved denominators are nonzero for all sufficiently small $|\alpha|$.

One can also try to eliminate the $w^2\bar{w}$-term by formally setting

$$h_{21} = \frac{g_{21}}{\lambda + \bar{\lambda}}.$$

One can also try to eliminate the $w^2\bar{w}$-term by formally setting

$$h_{21} = \frac{g_{21}}{\lambda + \bar{\lambda}}.$$

This is possible for small $\alpha \neq 0$, but the denominator vanishes at $\alpha = 0$: $\lambda(0) + \bar{\lambda}(0) = i\omega_0 - i\omega_0 = 0$. To obtain a transformation that is smoothly dependent on α, set $h_{21} = 0$, which results in

$$c_1 = \frac{g_{21}}{2}. \quad \square$$

Remark:

The remaining cubic $w^2\bar{w}$-term is called a *resonant term*. Note that its coefficient is the *same* as the coefficient of the cubic term $z^2\bar{z}$ in the original equation in Lemma 3.5. ♢

We now combine the two previous lemmas.

Lemma 3.6 (Poincaré normal form for the Hopf bifurcation) *The equation*

$$\dot{z} = \lambda z + \sum_{2 \le k+l \le 3} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l + O(|z|^4), \tag{3.15}$$

where $\lambda = \lambda(\alpha) = \mu(\alpha) + i\omega(\alpha), \mu(0) = 0, \omega(0) = \omega_0 > 0,$ *and* $g_{ij} = g_{ij}(\alpha)$, *can be transformed by an invertible parameter-dependent change of complex coordinate, smoothly depending on the parameter,*

$$\begin{aligned} z = w \quad &+ \quad \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2 \\ &+ \quad \frac{h_{30}}{6}w^3 + \frac{h_{12}}{2}w\bar{w}^2 + \frac{h_{03}}{6}\bar{w}^3, \end{aligned}$$

for all sufficiently small $|\alpha|$, *into an equation with only the resonant cubic term:*

$$\dot{w} = \lambda w + c_1 w^2 \bar{w} + O(|w|^4), \tag{3.16}$$

where $c_1 = c_1(\alpha)$.

Proof:

Obviously, a superposition of the transformations defined in Lemmas 3.4 and 3.5 does the job. First, perform the transformation

$$z = w + \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2, \tag{3.17}$$

with

$$h_{20} = \frac{g_{20}}{\lambda}, \ h_{11} = \frac{g_{11}}{\bar{\lambda}}, \ h_{02} = \frac{g_{02}}{2\bar{\lambda} - \lambda},$$

defined in Lemma 3.4. This annihilates all the quadratic terms but also with

$$h_{20} = \frac{g_{20}}{\lambda}, \; h_{11} = \frac{g_{11}}{\bar{\lambda}}, \; h_{02} = \frac{g_{02}}{2\bar{\lambda} - \lambda},$$

defined in Lemma 3.4. This annihilates all the quadratic terms but also changes the coefficients of the cubic terms. The coefficient of $w^2\bar{w}$ will be $\frac{1}{2}\tilde{g}_{21}$, say, instead of $\frac{1}{2}g_{21}$. Then make the transformation from Lemma 3.5 that eliminates all the cubic terms but the resonant one. The coefficient of this term remains $\frac{1}{2}\tilde{g}_{21}$. Since terms of order four and higher appearing in the superposition affect only $O(|w|^4)$ terms in (3.16), they can be truncated. □

Thus, all we need to compute to get the coefficient c_1 in terms of the given equation (3.15) is a new coefficient $\frac{1}{2}\tilde{g}_{21}$ of the $w^2\bar{w}$-term after the *quadratic* transformation (3.17). We can do this computation in the same manner as in Lemmas 3.4 and 3.5, namely, inverting (3.17). Unfortunately, now we have to know the inverse map up to and including *cubic* terms.[2] However, there is a possibility to avoid explicit inverting of (3.17).

Indeed, we can express $\dot{z}$ in terms of $w, \bar{w}$ in two ways. One way is to substitute (3.17) into the original equation (3.15). Alternatively, since we

---

[2]Actually, only the "resonant" cubic term of the inverse is required:

$$w = z - \frac{h_{20}}{2}z^2 - h_{11}z\bar{z} - \frac{h_{02}}{2}\bar{z}^2 + \frac{1}{2}(3h_{11}h_{20} + 2|h_{11}|^2 + |h_{02}|^2)z^2\bar{z} + \cdots,$$

where the dots now mean all undisplayed terms.

know the resulting form (3.16) to which (3.15) can be transformed, $\dot{z}$ can be computed by differentiating (3.17),

$$\dot{z} = \dot{w} + h_{20}w\dot{w} + h_{11}(w\dot{\bar{w}} + \bar{w}\dot{w}) + h_{02}\dot{\bar{w}},$$

and then by substituting $\dot{w}$ and its complex conjugate, using (3.16). Comparing the coefficients of the quadratic terms in the obtained expressions for $\dot{z}$ gives the above formulas for h_{20}, h_{11}, and h_{02}, while equating the coefficients in front of the $w|w|^2$-term leads to

$$c_1 = \frac{g_{20}g_{11}(2\lambda + \bar{\lambda})}{2|\lambda|^2} + \frac{|g_{11}|^2}{\lambda} + \frac{|g_{02}|^2}{2(2\lambda - \bar{\lambda})} + \frac{g_{21}}{2}.$$

This formula gives us the dependence of c_1 on α if we recall that λ and g_{ij} are smooth functions of the parameter. At the bifurcation parameter value $\alpha = 0$, the previous equation reduces to

$$c_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}. \qquad (3.18)$$

Now we want to transform the Poincaré normal form into the normal form studied in the previous section.

Lemma 3.7 *Consider the equation*

form studied in the previous section.

Lemma 3.7 *Consider the equation*

$$\frac{dw}{dt} = (\mu(\alpha) + i\omega(\alpha))w + c_1(\alpha)w|w|^2 + O(|w|^4),$$

where $\mu(0) = 0$, *and* $\omega(0) = \omega_0 > 0$.

Suppose $\mu'(0) \neq 0$ *and* $\text{Re } c_1(0) \neq 0$. *Then, the equation can be transformed by a parameter-dependent linear coordinate transformation, a time rescaling, and a nonlinear time reparametrization into an equation of the form*

$$\frac{du}{d\theta} = (\beta + i)u + su|u|^2 + O(|u|^4),$$

where u *is a new complex coordinate, and* θ, β *are the new time and parameter, respectively, and* $s = \text{sign Re } c_1(0) = \pm 1$.

Proof:

Step 1 (Linear time scaling). Introduce the new time $\tau = \omega(\alpha)t$. The time direction is preserved since $\omega(\alpha) > 0$ for all sufficiently small $|\alpha|$. Then,

$$\frac{dw}{d\tau} = (\beta + i)w + d_1(\beta)w|w|^2 + O(|w|^4),$$

where

$$\beta = \beta(\alpha) = \frac{\mu(\alpha)}{\omega(\alpha)}, \ d_1(\beta) = \frac{c_1(\alpha(\beta))}{\omega(\alpha(\beta))}.$$

We can consider β as a new parameter because

$$\beta(0) = 0, \ \beta'(0) = \frac{\mu'(0)}{\omega(0)} \neq 0,$$

and therefore the Inverse Function Theorem guarantees local existence and smoothness of α as a function of β. Notice that d_1 is *complex*.

Step 2 (*Nonlinear time reparametrization*). Change the time parametrization along the orbits by introducing a new time $\theta = \theta(\tau, \beta)$, where

$$d\theta = (1 + e_1(\beta)|w|^2) \ d\tau$$

with $e_1(\beta) = \text{Im } d_1(\beta)$. The time change is a near-identity transformation in a small neighborhood of the origin. Using the new definition of the time we obtain

$$\frac{dw}{d\theta} = (\beta + i)w + l_1(\beta)w|w|^2 + O(|w|^4),$$

where $l_1(\beta) = \text{Re } d_1(\beta) - \beta e_1(\beta)$ is *real* and

$$l_1(0) = \frac{\text{Re } c_1(0)}{\omega(0)}. \tag{3.19}$$

Step 3 (*Linear coordinate scaling*). Finally, introduce a new complex variable u:

$$w = \frac{u}{\sqrt{|l_1(\beta)|}},$$

which is possible due to Re $c_1(0) \neq 0$ and, thus, $l_1(0) \neq 0$. The equation able u:

$$w = \frac{u}{\sqrt{|l_1(\beta)|}},$$

which is possible due to Re $c_1(0) \neq 0$ and, thus, $l_1(0) \neq 0$. The equation then takes the required form:

$$\frac{du}{d\theta} = (\beta + i)u + \frac{l_1(\beta)}{|l_1(\beta)|}u|u|^2 + O(|u|^4) = (\beta + i)u + su|u|^2 + O(|u|^4),$$

with $s = \text{sign } l_1(0) = \text{sign Re } c_1(0)$. □

Definition 3.3 *The real function $l_1(\beta)$ is called the* first Lyapunov coefficient.

It follows from (3.19) that the first Lyapunov coefficient at $\beta = 0$ can be computed by the formula

$$l_1(0) = \frac{1}{2\omega_0^2} \text{Re}(ig_{20}g_{11} + \omega_0 g_{21}). \tag{3.20}$$

Thus, we need only certain second- and third-order derivatives of the right-hand sides at the bifurcation point to compute $l_1(0)$. Recall that the value of $l_1(0)$ does depend on the normalization of the eigenvectors q and p, while its sign (which is what only matters in the bifurcation analysis) is invariant under scaling of q, p obeying the relative normalization $\langle p, q \rangle = 1$. Notice that the equation of u with $s = -1$ written in real form coincides with system (3.10) from the previous section. We now summarize the obtained results in the following theorem.

Theorem 3.3 *Suppose a two-dimensional system*

$$\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1, \tag{3.21}$$

with smooth f, has for all sufficiently small $|\alpha|$ the equilibrium $x = 0$ with eigenvalues

$$\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha),$$

where $\mu(0) = 0, \ \omega(0) = \omega_0 > 0$.

Let the following conditions be satisfied:

(B.1) $l_1(0) \neq 0$, *where l_1 is the first Lyapunov coefficient*;
(B.2) $\mu'(0) \neq 0$.

Then, there are invertible coordinate and parameter changes and a time reparameterization transforming (3.21) *into*

$$\frac{d}{d\tau}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(\|y\|^4). \ \square$$

Using Lemma 3.2, we can drop the $O(\|y\|^4)$ terms and finally arrive at the following general result.

Theorem 3.4 (Topological normal form for the Hopf bifurcation) *Any generic two-dimensional, one-parameter system*

Theorem 3.4 (Topological normal form for the Hopf bifurcation) *Any generic two-dimensional, one-parameter system*

$$\dot{x} = f(x, \alpha),$$

having at $\alpha = 0$ *the equilibrium* $x = 0$ *with eigenvalues*

$$\lambda_{1,2}(0) = \pm i\omega_0, \ \ \omega_0 > 0,$$

is locally topologically equivalent near the origin to one of the following normal forms:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \ \square$$

Remark:

The genericity conditions assumed in Theorem 3.4 are the nondegeneracy condition (B.1) and the transversality condition (B.2) from Theorem 3.3. $\diamondsuit$

The preceding two theorems together with the normal form analysis of the previous section and formula (3.20) for $l_1(0)$ provide us with all the necessary tools for analysis of the Hopf bifurcation in generic two-dimensional systems. In Chapter 5 we will see how to deal with n-dimensional systems where $n > 2$.

Example 3.1 (Hopf bifurcation in a predator-prey model) Consider the following system of two differential equations:

$$
\begin{aligned}
\dot{x}_1 &= r x_1 (1 - x_1) - \frac{c x_1 x_2}{\alpha + x_1}, \\
\dot{x}_2 &= -d x_2 + \frac{c x_1 x_2}{\alpha + x_1}.
\end{aligned}
\tag{3.22}
$$

The system describes the dynamics of a simple predator-prey ecosystem (see, e.g., Holling [1965]). Here x_1 and x_2 are (scaled) population numbers, and r, c, d, and α are parameters characterizing the behavior of isolated populations and their interaction. Let us consider α as a control parameter and assume $c > d$.

To simplify calculations further, let us consider a *polynomial* system that has for $x_1 > -\alpha$ the same orbits as the original one (i.e., orbitally equivalent, see Chapter 2):

$$
\left\{
\begin{aligned}
\dot{x}_1 &= r x_1 (\alpha + x_1)(1 - x_1) - c x_1 x_2, \\
\dot{x}_2 &= -\alpha d x_2 + (c - d) x_1 x_2
\end{aligned}
\right.
\tag{3.23}
$$

(this system is obtained by multiplying both sides of the original system by the function $(\alpha + x_1)$ and introducing a new time variable τ by $dt = (\alpha + x_1)\, d\tau$).

(this system is obtained by multiplying both sides of the original system by the function $(\alpha + x_1)$ and introducing a new time variable τ by $dt = (\alpha + x_1)\, d\tau$).

System (3.23) has a nontrivial equilibrium

$$E_0 = \left(\frac{\alpha d}{c-d}, \frac{r\alpha}{c-d}\left[1 - \frac{\alpha d}{c-d}\right] \right).$$

The Jacobian matrix evaluated at this equilibrium is

$$A(\alpha) = \begin{pmatrix} \dfrac{\alpha r d(c+d)}{(c-d)^2}\left[\dfrac{c-d}{c+d} - \alpha\right] & -\dfrac{\alpha c d}{c-d} \\ \dfrac{\alpha r(c-d(1+\alpha))}{c-d} & 0 \end{pmatrix},$$

and thus

$$\mu(\alpha) = \frac{\sigma(\alpha)}{2} = \frac{\alpha r d(c+d)}{2(c-d)^2}\left[\frac{c-d}{c+d} - \alpha\right].$$

We have $\mu(\alpha_0) = 0$ for

$$\alpha_0 = \frac{c-d}{c+d}.$$

Moreover,

$$\omega^2(\alpha_0) = \frac{rc^2 d(c-d)}{(c+d)^3} > 0. \qquad (3.24)$$

Therefore, at $\alpha = \alpha_0$ the equilibrium E_0 has eigenvalues $\lambda_{1,2}(\alpha_0) = \pm i\omega(\alpha_0)$ and a Hopf bifurcation takes place.[3] The equilibrium is stable for $\alpha > \alpha_0$ and unstable for $\alpha < \alpha_0$. Notice that the critical value of α corresponds to the passing of the line defined by $\dot{x}_2 = 0$ through the maximum of the curve defined by $\dot{x}_1 = 0$ (see Figure 3.10). Thus, if the line $\dot{x}_2 = 0$ is to the

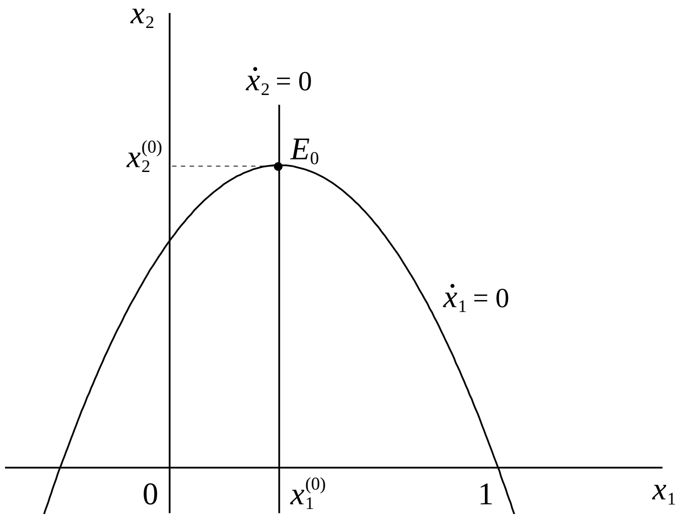


FIGURE 3.10. Zero-isoclines at the Hopf bifurcation.

right of the maximum, the point is stable, while if this line is to the left, the point is unstable. To apply the normal form theorem to the analysis of this Hopf bifurcation, we have to check whether the genericity conditions of Theorem 3.3 are satisfied. The transversality condition (B.2) is easy to right of the maximum, the point is stable, while if this line is to the left, the point is unstable. To apply the normal form theorem to the analysis of this Hopf bifurcation, we have to check whether the genericity conditions of Theorem 3.3 are satisfied. The transversality condition (B.2) is easy to verify:

$$\mu'(\alpha_0) = -\frac{\alpha_0 r d(c+d)}{2(c-d)^2} < 0.$$

To compute the first Lyapunov coefficient, fix the parameter α at its critical value α_0. At $\alpha = \alpha_0$, the nontrivial equilibrium E_0 at $\alpha = \alpha_0$ has the coordinates

$$x_1^{(0)} = \frac{d}{c+d}, \quad x_2^{(0)} = \frac{rc}{(c+d)^2}.$$

Translate the origin of the coordinates to this equilibrium by the change of variables

$$\begin{cases} x_1 & = & x_1^{(0)} + \xi_1, \\ x_2 & = & x_2^{(0)} + \xi_2. \end{cases}$$

This transforms system (3.23) into

$$\dot{\xi}_1 \quad = \quad -\frac{cd}{c+d}\xi_2 - \frac{rd}{c+d}\xi_1^2 - c\xi_1\xi_2 - r\xi_1^3 \;\equiv\; F_1(\xi_1, \xi_2),$$

---

[3] Since (3.23) is only orbitally equivalent to (3.22), the value of $\omega(\alpha_0)$ given by (3.24) *cannot* be used directly to evaluate the period of small oscillations around E_0 in the original system.

$$\dot{\xi}_2 = \frac{cr(c-d)}{(c+d)^2}\xi_1 + (c-d)\xi_1\xi_2 \equiv F_2(\xi_1, \xi_2).$$

This system can be represented as

$$\dot{\xi} = A\xi + \frac{1}{2}B(\xi, \xi) + \frac{1}{6}C(\xi, \xi, \xi),$$

where $A = A(\alpha_0)$, and the multilinear functions B and C take on the planar vectors $\xi = (\xi_1, \xi_2)^T$, $\eta = (\eta_1, \eta_2)^T$, and $\zeta = (\zeta_1, \zeta_2)^T$ the values

$$B(\xi, \eta) = \begin{pmatrix} -\frac{2rd}{(c+d)}\xi_1\eta_1 - c(\xi_1\eta_2 + \xi_2\eta_1) \\ (c-d)(\xi_1\eta_2 + \xi_2\eta_1) \end{pmatrix}$$

and

$$C(\xi, \eta, \zeta) = \begin{pmatrix} -6r\xi_1\eta_1\zeta_1 \\ 0 \end{pmatrix}.$$

Write the matrix $A(\alpha_0)$ in the form

$$A = \begin{pmatrix} 0 & -\frac{cd}{c+d} \\ \frac{\omega^2(c+d)}{cd} & 0 \end{pmatrix},$$

where ω^2 is given by formula (3.24).[4] Now it is easy to check that complex vectors

$$q \sim \begin{pmatrix} cd \\ -i\omega(c+d) \end{pmatrix}, \quad p \sim \begin{pmatrix} \omega(c+d) \\ -icd \end{pmatrix},$$

where ω^2 is given by formula (3.24).[4] Now it is easy to check that complex vectors

$$q \sim \begin{pmatrix} cd \\ -i\omega(c+d) \end{pmatrix}, \quad p \sim \begin{pmatrix} \omega(c+d) \\ -icd \end{pmatrix},$$

are proper eigenvectors:

$$Aq = i\omega q, \quad A^T p = -i\omega p.$$

To achieve the necessary normalization $\langle p, q \rangle = 1$, we can take, for example,

$$q = \begin{pmatrix} cd \\ -i\omega(c+d) \end{pmatrix}, \quad p = \frac{1}{2\omega cd(c+d)} \begin{pmatrix} \omega(c+d) \\ -icd \end{pmatrix}.$$

The hardest part of the job is done, and now we can simply calculate[5]

$$g_{20} = \langle p, B(q,q) \rangle = \frac{cd(c^2 - d^2 - rd) + i\omega c(c+d)^2}{(c+d)},$$

---

[4]It is always useful to express the Jacobian matrix using ω, since this simplifies expressions for the eigenvectors.

[5]Another way to compute g_{20}, g_{11}, and g_{21} (which may be simpler if we use a symbolic manipulation software) is to define the complex-valued function

$$G(z,w) = \bar{p}_1 F_1(zq_1 + w\bar{q}_1, zq_2 + w\bar{q}_2) + \bar{p}_2 F_2(zq_1 + w\bar{q}_1, zq_2 + w\bar{q}_2),$$

where p, q are given above, and to evaluate its formal partial derivatives with respect to z, w at $z = w = 0$, obtaining $g_{20} = G_{zz}$, $g_{11} = G_{zw}$, and $g_{21} = G_{zzw}$. In this way no multilinear functions are necessary. See Exercise 4.

$$g_{11} = \langle p, B(q, \bar{q}) \rangle = -\frac{rcd^2}{(c+d)}, \quad g_{21} = \langle p, C(q, q, \bar{q}) \rangle = -3rc^2d^2,$$

and compute the first Lyapunov coefficient by formula (3.20),

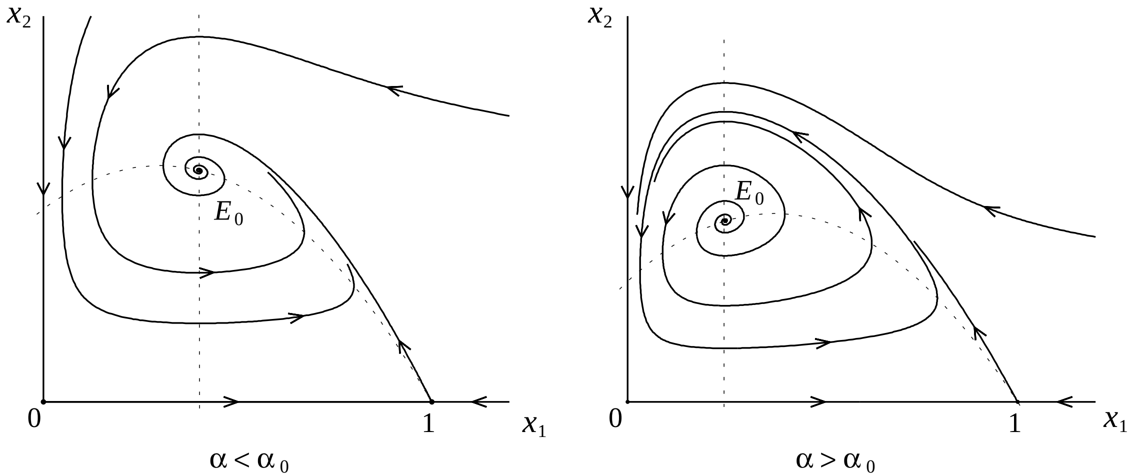


FIGURE 3.11. Hopf bifurcation in the predator-prey model.

$$l_1(\alpha_0) = \frac{1}{2\omega^2} \operatorname{Re}(ig_{20}g_{11} + \omega g_{21}) = -\frac{rc^2d^2}{\omega} < 0.$$

It is clear that $l_1(\alpha_0) < 0$ for all combinations of the fixed parameters. Thus, the nondegeneracy condition (B.1) of Theorem 3.3 holds as well. Therefore, a unique and stable limit cycle bifurcates from the equilibrium

It is clear that $l_1(\alpha_0) < 0$ for all combinations of the fixed parameters. Thus, the nondegeneracy condition (B.1) of Theorem 3.3 holds as well. Therefore, a unique and stable limit cycle bifurcates from the equilibrium via the Hopf bifurcation for $\alpha < \alpha_0$ (see Figure 3.11). ◇

3.6 Exercises

(1) (Fold bifurcation in ecology) Consider the following differential equation, which models a single population under a constant harvest:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \alpha,$$

where x is the population number; r and K are the *intrinsic growth rate* and the *carrying capacity* of the population, respectively, and α is the *harvest rate*, which is a control parameter. Find a parameter value α_0 at which the system has a fold bifurcation, and check the genericity conditions of Theorem 3.1. Based on the analysis, explain what might be a result of overharvesting on the ecosystem dynamics. Is the bifurcation catastrophic in this example?

(2) (Complex notation) Verify that

$$\dot{z} = iz + (i+1)z^2 + 2iz\bar{z} + (i-1)\bar{z}^2$$

is a complex form of the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{6}{\sqrt{3}} \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix},$$

provided that the eigenvectors are selected in the form

$$q = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 \\ -1+i \end{pmatrix}, \quad p = \frac{3}{2\sqrt{3}} \begin{pmatrix} 1+i \\ 2i \end{pmatrix}.$$

How will the complex form change if one instead adopts a different setting of q, p satisfying $\langle p, q \rangle = 1$?

(3) (Nonlinear stability) Write the system

$$\begin{cases} \dot{x} &= -y - xy + 2y^2, \\ \dot{y} &= x - x^2 y, \end{cases}$$

in terms of the complex coordinate $z = x + iy$, and compute the normal form coefficient $c_1(0)$ by formula (3.18). Is the origin stable?

(4) (Hopf bifurcation in the Brusselator) Consider the Brusselator system (1.8) from Chapter 1:

$$\begin{cases} \dot{x}_1 &= A - (B+1)x_1 + x_1^2 x_2 \equiv F_1(x_1, x_2, A, B), \end{cases}$$

system (1.8) from Chapter 1:

$$\begin{cases} \dot{x}_1 &= A - (B+1)x_1 + x_1^2 x_2 \equiv F_1(x_1, x_2, A, B), \\ \dot{x}_2 &= Bx_1 - x_1^2 x_2 \equiv F_2(x_1, x_2, A, B). \end{cases}$$

Fix $A > 0$ and take B as a bifurcation parameter. Using one of the available computer algebra systems, prove that at $B = 1 + A^2$ the system exhibits a supercritical Hopf bifurcation.

(*Hint:* The following sequence of MAPLE commands solves the problem:

```
> with(linalg);
> readlib(mtaylor);
> readlib(coeftayl);
```

The first command above allows us to use the MAPLE linear algebra package. The other two commands load the procedures `mtaylor` and `coeftayl`, which compute the truncated multivariate Taylor series expansion and its individual coefficients, respectively, from the MAPLE library.

```
> F[1]:=A-(B+1)*X[1]+X[1] 2*X[2];
> F[2]:=B*X[1]-X[1] 2*X[2];
> J:=jacobian([F[1],F[2]],[X[1],X[2]]);
> K:=transpose(J);
```

By these commands we input the right-hand sides of the system into MAPLE and compute the Jacobian matrix and its transpose.

```
> sol:=solve({F[1]=0,F[2]=0,trace(J)=0},{X[1],X[2],B});
> assign(sol);
> assume(A>0);
> omega:=sqrt(det(J));
```

These commands solve the following system of equations

$$\begin{cases} F(x_1, x_2, A, B) = 0, \\ \mathrm{tr}\ F_x(x_1, x_2, A, B) = 0, \end{cases}$$

for (x_1, x_2, B) and allow us to check that $\det F_x = A^2 > 0$ at the found solution. Thus, at $B = 1 + A^2$ the Brusselator has the equilibrium

$$X = \left(A, \frac{1 + A^2}{A} \right)^T$$

with purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega,\ \omega = A > 0$.

```
> ev:=eigenvects(J,'radical');
> q:=ev[1][3][1];
> et:=eigenvects(K,'radical');
> P:=et[2][3][1];
```

These commands show that

```
> P:=et[2][3][1];
```

These commands show that

$$q = \left(-\frac{iA + A^2}{1 + A^2},\ 1\right)^T,\quad p = \left(\frac{-iA + A^2}{A^2},\ 1\right)^T,$$

are the critical eigenvectors[6] of the Jacobian matrix $J = F_x$ and its transpose,

$$Jq = i\omega q,\ \ J^T p = -i\omega p.$$

```
> s1:=simplify(evalc(conjugate(P[1])*q[1]+conjugate(P[2]*q[2]));
> c:=simplify(evalc(1/conjugate(s1)));
> p[1]:=simplify(evalc(c*P[1]);
> p[2]:=simplify(evalc(c*P[2]);
> simplify(evalc(conjugate(p[1])*q[1]+conjugate(p[2])*q[2]));
```

By the commands above, we achieve the normalization $\langle p, q\rangle = 1$, finally taking

$$q = \left(-\frac{iA + A^2}{1 + A^2},\ 1\right)^T,\quad p = \left(-\frac{i(1 + A^2)}{2A},\ \frac{1 - iA}{2}\right)^T.$$

[6] Some implementations of MAPLE may produce the eigenvectors in a different form.

```
> F[1]:=A-(B+1)*x[1]+x[1] 2*x[2];
> F[2]:=B*x[1]-x[1] 2*x[2];
> x[1]:=evalc(X[1]+z*q[1]+z1*conjugate(q[1]));
> x[2]:=evalc(X[2]+z*q[2]+z1*conjugate(q[2]));
> H:=simplify(evalc(conjugate(p[1])*F[1]+conjugate(p[2])*F[2]))
```

By means of these commands, we compose $x = X + zq + \bar{z}\bar{q}$ and evaluate the function

$$H(z, \bar{z}) = \langle p, F(X + zq + \bar{z}\bar{q}, A, 1 + A^2) \rangle.$$

(In the MAPLE commands, `z1` stands for $\bar{z}$.)

```
> g[2,0]:=simplify(2*evalc(coeftayl(H,[z,z1]=[0,0],[2,0])));
> g[1,1]:=simplify(evalc(coeftayl(H,[z,z1]=[0,0],[1,1])));
> g[2,1]:=simplify(2*evalc(coeftayl(H,[z,z1]=[0,0],[2,1])));
> x[2]:=evalc(X[2]+z*q[2]+z1*conjugate(q[2]));
```

The above commands compute the needed Taylor expansion of $H(z, \bar{z})$ at $(z, \bar{z}) = (0, 0)$,

$$H(z, \bar{z}) = i\omega z + \sum_{2 \leq j+k \leq 3} \frac{1}{j!k!} g_{jk} z^k \bar{z}^k + O(|z|^4),$$

giving

$2\leq j+k\leq 3$

giving

$$g_{20}=A-i,\quad g_{11}=\frac{(A-i)(A^2-1)}{1+A^2},\quad g_{21}=-\frac{A(3A-i)}{1+A^2}.$$

```
> l[1]:=factor(1/(2*omega 2)*Re(I*g[2,0]*g[1,1]+omega*g{2,1]));
```

This final command computes the first Lyapunov coefficient

$$l_1=\frac{1}{2\omega^2}\mathrm{Re}(ig_{20}g_{11}+\omega g_{2,1})=-\frac{2+A^2}{2A(1+A^2)}<0,$$

and allows us to check that it is negative.)

(5) Check that each of the following systems has an equilibrium that exhibits the Hopf bifurcation at some value of α, and compute the first Lyapunov coefficient:

(a) *Rayleigh's equation*:

$$\ddot{x}+\dot{x}^3-2\alpha\dot{x}+x=0;$$

(*Hint:* Introduce $y=\dot{x}$ and rewrite the equation as a system of two differential equations.)

(b) *Van der Pol's oscillator*:

$$\ddot{y}-(\alpha-y^2)\dot{y}+y=0;$$

(c) *Bautin's example*:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + \alpha y + x^2 + xy + y^2; \end{cases}$$

(d) *Advertising diffusion model* [Feichtinger 1992]:

$$\begin{cases} \dot{x}_1 &= \alpha[1 - x_1 x_2^2 + A(x_2 - 1)], \\ \dot{x}_2 &= x_1 x_2^2 - x_2. \end{cases}$$

(6) Suppose that a system at the critical parameter values corresponding to the Hopf bifurcation has the form

$$\begin{aligned} \dot{x} &= -\omega y + \frac{1}{2} f_{xx} x^2 + f_{xy} xy + \frac{1}{2} f_{yy} y^2 \\ &\quad + \frac{1}{6} f_{xxx} x^3 + \frac{1}{2} f_{xxy} x^2 y + \frac{1}{2} f_{xyy} xy^2 + \frac{1}{6} f_{yyy} y^3 + \cdots, \\ \dot{y} &= \omega x + \frac{1}{2} g_{xx} x^2 + g_{xy} xy + \frac{1}{2} g_{yy} y^2 \\ &\quad + \frac{1}{6} g_{xxx} x^3 + \frac{1}{2} g_{xxy} x^2 y + \frac{1}{2} g_{xyy} xy^2 + \frac{1}{6} g_{yyy} y^3 + \cdots. \end{aligned}$$

Compute Re $c_1(0)$ in terms of the f's and g's. (*Hint:* See Guckenheimer & Holmes [1983, p. 156]. To apply the resulting formula, one needs to transform the system explicitly into its eigenbasis, which can always be

Compute $\mathrm{Re}\, c_1(0)$ in terms of the f's and g's. (*Hint:* See Guckenheimer & Holmes [1983, p. 156]. To apply the resulting formula, one needs to transform the system explicitly into its eigenbasis, which can always be avoided by using eigenvectors and complex notation, as described in this chapter.)

3.7 Appendix 1: Proof of Lemma 3.2

The following statement, which is Lemma 3.2 rewritten in complex form, will be proved in this appendix.

Lemma 3.8 *The system*

$$\dot{z} = (\alpha + i)z - z|z|^2 + O(|z|^4) \tag{A.1}$$

is locally topologically equivalent near the origin to the system

$$\dot{z} = (\alpha + i)z - z|z|^2. \tag{A.2}$$

Proof:
Step 1 (*Existence and uniqueness of the cycle*). Write system (A.1) in polar coordinates (ρ, φ):

$$\begin{cases} \dot{\rho} &= \rho(\alpha - \rho^2) + \Phi(\rho, \varphi), \\ \dot{\varphi} &= 1 + \Psi(\rho, \varphi), \end{cases} \tag{A.3}$$

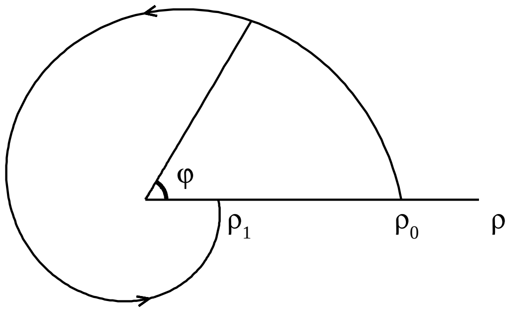


FIGURE 3.12. Poincaré map for the Hopf bifurcation.

where $\Phi = O(|\rho|^4), \Psi = O(|\rho|^3)$, and the α-dependence of these functions is not indicated to simplify notations. An orbit of $(A.3)$ starting at $(\rho, \varphi) = (\rho_0, 0)$ has the following representation (see Figure 3.12): $\rho = \rho(\varphi; \rho_0), \rho_0 = \rho(0; \rho_0)$ with ρ satisfying the equation

$$\frac{d\rho}{d\varphi} = \frac{\rho(\alpha - \rho^2) + \Phi}{1 + \Psi} = \rho(\alpha - \rho^2) + R(\rho, \varphi), \tag{A.4}$$

where $R = O(|\rho|^4)$. Notice that the transition from (A.3) to (A.4) is equivalent to the introduction of a new time parametrization in which $\dot{\varphi} = 1$, which implies that the return time to the half-axis $\varphi = 0$ is the same for all orbits starting on this axis with $\rho_0 > 0$. Since $\rho(\varphi; 0) \equiv 0$, we can write the Taylor expansion for $\rho(\varphi; \rho_0)$,

$$\rho = u_1(\varphi)\rho_0 + u_2(\varphi)\rho_0^2 + u_3(\varphi)\rho_0^3 + O(|\rho_0|^4), \tag{A.5}$$

all orbits starting on this axis with $\rho_0 > 0$. Since $\rho(\varphi, 0) = 0$, we can write the Taylor expansion for $\rho(\varphi; \rho_0)$,

$$\rho = u_1(\varphi)\rho_0 + u_2(\varphi)\rho_0^2 + u_3(\varphi)\rho_0^3 + O(|\rho_0|^4). \tag{A.5}$$

Substituting (A.5) into (A.4) and solving the resulting linear differential equations at corresponding powers of ρ_0 with initial conditions $u_1(0) = 1, u_2(0) = u_3(0) = 0$, we get

$$u_1(\varphi) = e^{\alpha\varphi},\ u_2(\varphi) \equiv 0,\ u_3(\varphi) = e^{\alpha\varphi}\frac{1 - e^{2\alpha\varphi}}{2\alpha}.$$

Notice that these expressions are *independent* of the term $R(\rho, \varphi)$. Therefore, the return map $\rho_0 \mapsto \rho_1 = \rho(2\pi, \rho_0)$ has the form

$$\rho_1 = e^{2\pi\alpha}\rho_0 - e^{2\pi\alpha}[2\pi + O(\alpha)]\rho_0^3 + O(\rho_0^4) \tag{A.6}$$

for *all* $R = O(\rho^4)$. The map (A.6) can easily be analyzed for sufficiently small ρ_0 and $|\alpha|$. There is a neighborhood of the origin in which the map has only a trivial fixed point for small $\alpha < 0$ and an extra fixed point, $\rho_0^{(0)} = \sqrt{\alpha} + \cdots$, for small $\alpha > 0$ (see Figure 3.13). The stability of the fixed points is also easily obtained from (A.6). Taking into account that a positive fixed point of the map corresponds to a limit cycle of the system, we can conclude that system (A.3) (or (A.1)) with any $O(|z|^4)$ terms has a unique (stable) limit cycle bifurcating from the origin and existing for $\alpha > 0$ as in system (A.2). Therefore, in other words, higher-order terms do

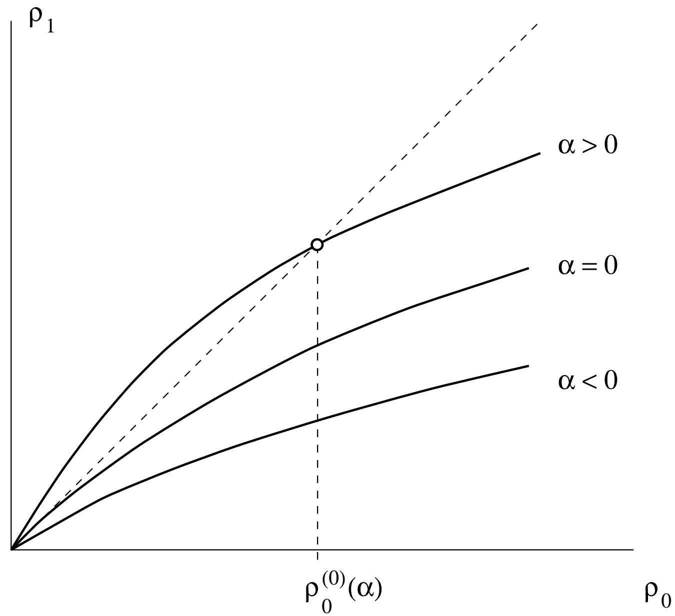


FIGURE 3.13. Fixed point of the return map.

not affect the limit cycle bifurcation in some neighborhood of $z = 0$ for $|\alpha|$ sufficiently small.

Step 2 (*Construction of a homeomorphism*). The established existence and uniqueness of the limit cycle is enough for all applications. Nevertheless, extra work must be done to prove the topological equivalence of the phase portraits.

uniqueness of the limit cycle is enough for all applications. Nevertheless, extra work must be done to prove the topological equivalence of the phase portraits.

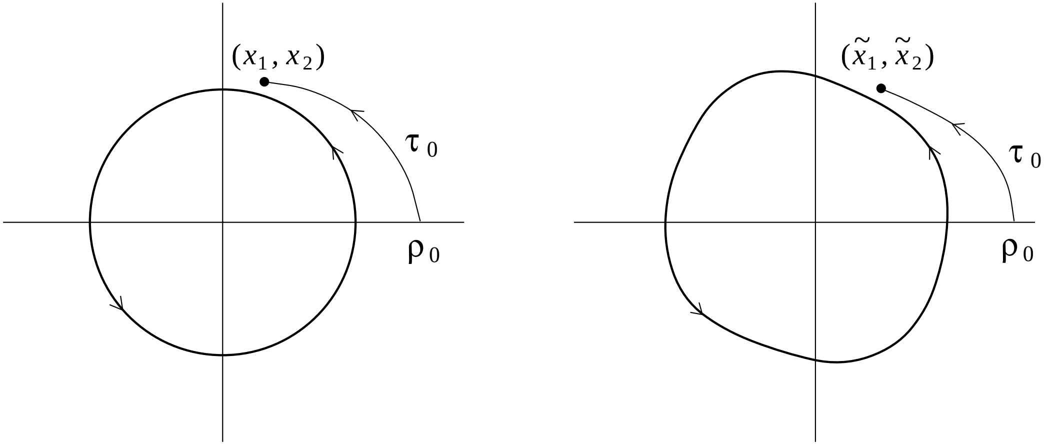


FIGURE 3.14. Construction of the homeomorphism near the Hopf bifurcation.

Fix α small but positive. Both systems (A.1) and (A.2) have a limit cycle in some neighborhood of the origin. Assume that the time reparametrization resulting in the constant return time 2π is performed in system (A.1) (see the previous step). Also, apply a linear scaling of the coordinates in system (A.1) such that the point of intersection of the cycle and the horizontal half-axis is at $x_1 = \sqrt{\alpha}$.

Define a map $z \mapsto \tilde{z}$ by the following construction. Take a point $z = x_1 + ix_2$ and find values (ρ_0, τ_0), where τ_0 is the minimal time required for an orbit of system (A.2) to approach the point z starting from the horizontal half-axis with $\rho = \rho_0$. Now, take the point on this axis with $\rho = \rho_0$ and construct an orbit of system (A.1) on the time interval $[0, \tau_0]$ starting at this point. Denote the resulting point by $\tilde{z} = \tilde{x}_1 + i\tilde{x}_2$ (see Figure 3.14). Set $\tilde{z} = 0$ for $z = 0$.

The map constructed is a homeomorphism that, for $\alpha > 0$, maps orbits of system (A.2) in some neighborhood of the origin into orbits of (A.1) preserving time direction. The case $\alpha < 0$ can be considered in the same way without rescaling the coordinates. □

3.8 Appendix 2: Bibliographical notes

The fold bifurcation of equilibria has essentially been known for centuries. Since any scalar system can be written as $\dot{x} = -\psi_x(x, \alpha)$, for some function ψ, results on the classification of generic parameter-dependent *gradient systems* from catastrophe theory are relevant. Thus, the topological normal form for the fold bifurcation appeared in the list of seven elementary catastrophes by Thom [1972]. Actually, there are many interconnections between bifurcation theory of dynamical systems and singularity theory of smooth functions. The books by Poston & Stewart [1978] and Arnold [1984] are recommended as an introduction to this latter subject. It should be noticed, however, that most results from singularity theory are directly applicable to the analysis of equilibria but not to the analysis of phase of smooth functions. The books by Poston & Stewart [1978] and Arnold [1984] are recommended as an introduction to this latter subject. It should be noticed, however, that most results from singularity theory are directly applicable to the analysis of equilibria but not to the analysis of phase portraits.

The normalization technique used in the analysis of limit cycle bifurcations was developed by Poincaré [1879]. A general presentation of the theory of normal forms can be found in Arnold [1983], Guckenheimer & Holmes [1983], and Vanderbauwhede [1989], where it is also explained how to apply this theory to local bifurcation problems. Actually, for the limit cycle bifurcation analysis only a small part of this theory is really required. Theorem 3.4 was stated and briefly proved by Arnold [1972, 1983]. We follow his approach.

Phase-portrait bifurcations in a generic one-parameter system on the plane near an equilibrium with purely imaginary eigenvalues was studied first by Andronov & Leontovich [1939]. They used a *succession function* (return map) technique originally due to Lyapunov [1892] without benefiting from the normalization. An explicit expression for the first Lyapunov coefficient in terms of Taylor coefficients of a general planar system was obtained by Bautin [1949]. An exposition of the results by Andronov and his co-workers on this bifurcation can be found in Andronov et al. [1973].

Hopf [1942] proved the appearance of a family of periodic solutions of increasing amplitude in n-dimensional systems having an equilibrium with a pair of purely imaginary eigenvalues at some critical parameter value. He did not consider bifurcations of the whole phase portrait. An Englishlanguage translation of Hopf's paper is included in Marsden & McCracken [1976]. This very useful book also contains a derivation of the first Lyapunov coefficient and a proof of Hopf's result based on the Implicit Function Theorem.

A much simpler derivation of the Lyapunov coefficient (actually, c_1) is given by Hassard, Kazarinoff & Wan [1981] using the complex form of the Poincaré normalization. We essentially use their technique, although we do not assume that the Poincaré normal form is known a priori. Formulas to compute Taylor coefficients of the complex equation without an intermediate transformation of the system into its eigenbasis can also be extracted from their book (applying the center manifold reduction technique to the trivial planar case; see Chapter 5). We also extensively use time reparametrization to obtain a simpler normal form, which is then used to prove existence and uniqueness of the cycle and in the analysis of the whole phase-portrait bifurcations (see Appendix 1).

There exist other approaches to prove the generation of periodic solutions under the Hopf conditions. An elegant one is to reformulate the problem as that of finding a family of solutions of an abstract equation in a functional space of periodic functions and to apply the *Lyapunov-Schmidt reduction*. This approach, allowing a generalization to infinite-dimensional dynamical systems, is far beyond the scope of this book (see, e.g., Chow & Hale [1982] or Iooss & Joseph [1980]).

or Iooss & Joseph [1980]).

4

One-Parameter Bifurcations of Fixed Points in Discrete-Time Dynamical Systems

In this chapter, which is organized very much like Chapter 3, we present bifurcation conditions defining the simplest bifurcations of fixed points in n-dimensional discrete-time dynamical systems: the fold, the flip, and the Neimark-Sacker bifurcations. Then we study these bifurcations in the lowbifurcation conditions defining the simplest bifurcations of fixed points in n-dimensional discrete-time dynamical systems: the fold, the flip, and the Neimark-Sacker bifurcations. Then we study these bifurcations in the lowest possible dimension in which they can occur: the fold and flip bifurcations for scalar systems and the Neimark-Sacker bifurcation for planar systems. In Chapter 5 it will be shown how to apply these results to n-dimensional systems when n is larger than one or two, respectively.

4.1 Simplest bifurcation conditions

Consider a discrete-time dynamical system depending on a parameter

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^1,$$

where the map f is smooth with respect to both x and α. Sometimes we will write this system as

$$\tilde{x} = f(x, \alpha), \quad x, \tilde{x} \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^1,$$

where $\tilde{x}$ denotes the image of x under the action of the map. Let $x = x_0$ be a hyperbolic fixed point of the system for $\alpha = \alpha_0$. Let us monitor this fixed point and its multipliers while the parameter varies. It is clear that there are, generically, only three ways in which the hyperbolicity condition can be violated. Either a simple positive multiplier approaches the unit circle

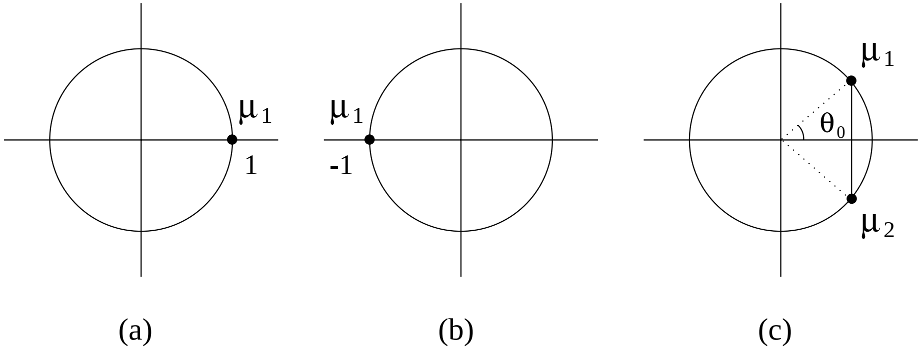


FIGURE 4.1. Codim 1 critical cases.

and we have $\mu_1 = 1$ (see Figure 4.1(a)), or a simple negative multiplier approaches the unit circle and we have $\mu_1 = -1$ (Figure 4.1(b)), or a pair of simple complex multipliers reaches the unit circle and we have $\mu_{1,2} = e^{\pm i\theta_0}, 0 < \theta_0 < \pi$ (Figure 4.1(c)), for some value of the parameter. It is obvious that one needs more parameters to allocate extra eigenvalues on the unit circle.

The rest of the chapter is devoted to the proof that a nonhyperbolic fixed point satisfying one of the above conditions is *structurally unstable*, and to the analysis of the corresponding bifurcations of the local phase portrait under variation of the parameter. Let us finish this section with the following definitions, the reasoning for which will become clear later.

Definition 4.1 *The bifurcation associated with the appearance of $\mu_1 = 1$ is called a* fold *(or tangent) bifurcation.*

the following definitions, the reasoning for which will become clear later.

Definition 4.1 *The bifurcation associated with the appearance of* $\mu_1 = 1$ *is called a* fold (*or* tangent) *bifurcation.*

Remark:

This bifurcation is also referred to as a *limit point, saddle-node bifurcation, turning point,* among others. ♢

Definition 4.2 *The bifurcation associated with the appearance of* $\mu_1 = -1$ *is called a* flip (*or* period-doubling) *bifurcation.*

Definition 4.3 *The bifurcation corresponding to the presence of* $\mu_{1,2} = e^{\pm i\theta_0}, 0 < \theta_0 < \pi$, *is called a* Neimark-Sacker (*or* torus) *bifurcation.*

Notice that the fold and flip bifurcations are possible if $n \geq 1$, but for the Neimark-Sacker bifurcation we need $n \geq 2$.

4.2 The normal form of the fold bifurcation

Consider the following one-dimensional dynamical system depending on one parameter:

$$x \mapsto \alpha + x + x^2 \equiv f(x, \alpha) \equiv f_\alpha(x). \tag{4.1}$$

The map f_α is invertible for $|\alpha|$ small in a neighborhood of the origin. The

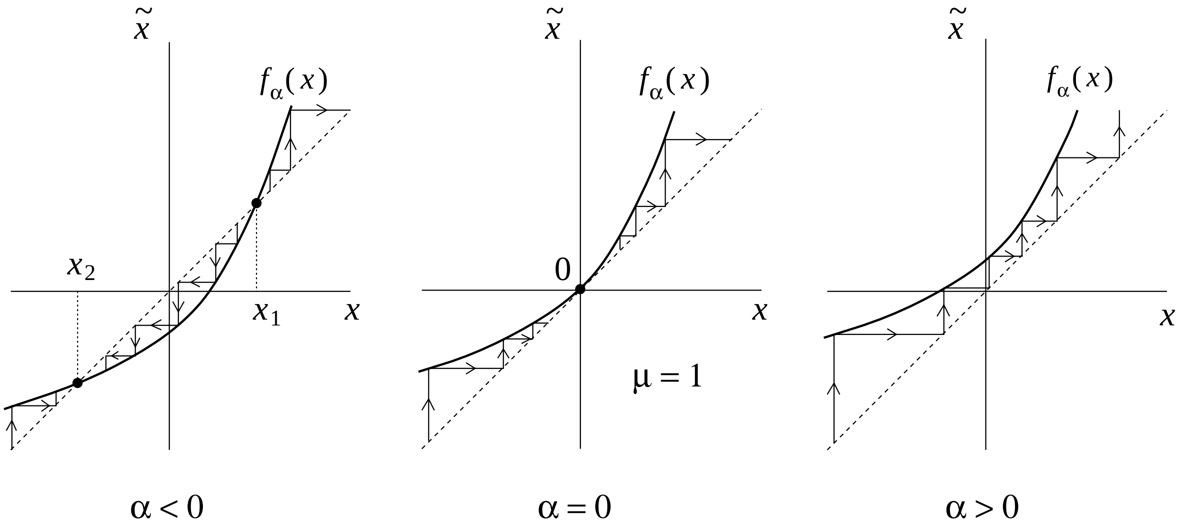


FIGURE 4.2. Fold bifurcation.

system has at $\alpha = 0$ a nonhyperbolic fixed point $x_0 = 0$ with $\mu = f_x(0,0) = 1$. The behavior of the system near $x = 0$ for small $|\alpha|$ is shown in Figure 4.2. For $\alpha < 0$ there are two fixed points in the system: $x_{1,2}(\alpha) = \pm\sqrt{-\alpha}$, the left of which is stable, while the right one is unstable. For $\alpha > 0$ there are no fixed points in the system. While α crosses zero from negative to positive values, the two fixed points (stable and unstable) "collide," forming at $\alpha = 0$ a fixed point with $\mu = 1$, and disappear. This is a fold (tangent) bifurcation in the discrete-time dynamical system.

There is, as usual, another way of presenting this bifurcation: plotting at $\alpha = 0$ a fixed point with $\mu = 1$, and disappear. This is a fold (tangent) bifurcation in the discrete-time dynamical system.

There is, as usual, another way of presenting this bifurcation: plotting a bifurcation diagram in the direct product of the phase and parameter spaces, namely, in the (x, α)-plane. The *fixed-point manifold* $x - f(x, \alpha) = 0$ is simply the parabola $\alpha = -x^2$ (see Figure 4.3). Fixing some α, we can easily determine the number of fixed points in the system for this parameter value. At $(x, \alpha) = (0, 0)$ a map projecting the fixed-point manifold onto the α-axis has a singularity of the fold type.

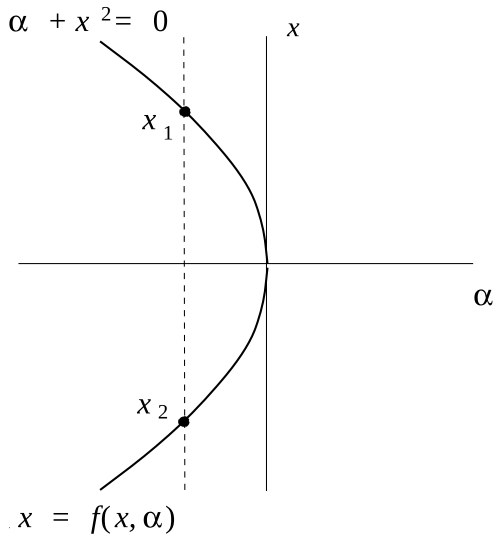


FIGURE 4.3. Fixed point manifold.

Remark:

The system $x \mapsto \alpha + x - x^2$ can be considered in the same way. The analysis reveals two fixed points appearing for $\alpha > 0$. $\diamondsuit$

Now add higher-order terms to system (4.1), i.e., consider the system

$$x \mapsto \alpha + x + x^2 + x^3\psi(x, \alpha) \equiv F_\alpha(x), \tag{4.2}$$

where $\psi = \psi(x, \alpha)$ depends smoothly on (x, α). It is easy to check that in a sufficiently small neighborhood of $x = 0$ the number and the stability of the fixed points are the same for system (4.2) as for system (4.1) at corresponding parameter values, provided $|\alpha|$ is small enough. Moreover, a homeomorphism h_α of a neighborhood of the origin mapping orbits of (4.1) into the corresponding orbits of (4.2) can be constructed for each small $|\alpha|$. This property was called *local topological equivalence* of parameter-dependent systems in Chapter 2. It should be noted that construction of h_α is not as simple as in the continuous-time case (cf. Lemma 3.1). In the present case, a homeomorphism mapping the fixed points of (4.1) into the corresponding fixed points of (4.2) will not necessarily map other orbits of (4.1) into orbits of (4.2). Nevertheless, a homeomorphism h_α satisfying the condition

$$f_\alpha(x) = h_\alpha^{-1}(F_\alpha(h_\alpha(x)))$$

for all (x, α) in a neighbourghood of $(0, 0)$ (cf. Chapter 2) exists. Thus, the following lemma holds.

$$f_\alpha(x) = h_\alpha^{-1}(F_\alpha(h_\alpha(x)))$$

for all (x, α) in a neighbourghood of $(0, 0)$ (cf. Chapter 2) exists. Thus, the following lemma holds.

Lemma 4.1 *The system*

$$x \mapsto \alpha + x + x^2 + O(x^3)$$

is locally topologically equivalent near the origin to the system

$$x \mapsto \alpha + x + x^2. \ \square$$

4.3 Generic fold bifurcation

We shall show that system (4.1) (with a possible sign change of the term x^2) is a topological normal form of a generic one-dimensional discrete-time system having a fold bifurcation. In Chapter 5 we will also see that in some strong sense it describes the fold bifurcation in a generic n-dimensional system.

Theorem 4.1 *Suppose that a one-dimensional system*

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}, \ \alpha \in \mathbb{R}^1, \tag{4.3}$$

with smooth f, has at $\alpha = 0$ the fixed point $x_0 = 0$, and let $\mu = f_x(0, 0) = 1$. Assume that the following conditions are satisfied:

(A.1) $f_{xx}(0,0) \neq 0$;
(A.2) $f_\alpha(0,0) \neq 0$.

Then there are smooth invertible coordinate and parameter changes transforming the system into

$$\eta \mapsto \beta + \eta \pm \eta^2 + O(\eta^3).$$

Proof:

Expand $f(x,\alpha)$ in a Taylor series with respect to x at $x = 0$:

$$f(x,\alpha) = f_0(\alpha) + f_1(\alpha)x + f_2(\alpha)x^2 + O(x^3).$$

Two conditions are satisfied: $f_0(0) = f(0,0) = 0$ (*fixed-point condition*) and $f_1(0) = f_x(0,0) = 1$ (*fold bifurcation condition*). Since $f_1(0) = 1$, we may write

$$f(x,\alpha) = f_0(\alpha) + [1 + g(\alpha)]x + f_2(\alpha)x^2 + O(x^3),$$

where $g(\alpha)$ is smooth and $g(0) = 0$.

As in the proof of Theorem 3.1 in Chapter 3, perform a coordinate shift by introducing a new variable ξ:

$$\xi = x + \delta, \tag{4.4}$$

where $\delta = \delta(\alpha)$ is to be defined suitably. The transformation (4.4) yields

$$\xi = x + \delta, \tag{4.4}$$

where $\delta = \delta(\alpha)$ is to be defined suitably. The transformation (4.4) yields

$$\tilde{\xi} = \tilde{x} + \delta = f(x, \alpha) + \delta = f(\xi - \delta, \alpha) + \delta.$$

Therefore,

$$\begin{aligned} \tilde{\xi} \quad = \quad & [f_0(\alpha) - g(\alpha)\delta + f_2(\alpha)\delta^2 + O(\delta^3)] \\ & + \xi + [g(\alpha) - 2f_2(\alpha)\delta + O(\delta^2)]\xi \\ & + [f_2(\alpha) + O(\delta)]\xi^2 + O(\xi^3). \end{aligned}$$

Assume that

(A.1) $$f_2(0) = \tfrac{1}{2} f_{xx}(0, 0) \neq 0.$$

Then there is a smooth function $\delta(\alpha)$, which annihilates the parameter-dependent linear term in the above map for all sufficiently small $|\alpha|$. Indeed, the condition for that term to vanish can be written as

$$F(\alpha, \delta) \equiv g(\alpha) - 2f_2(\alpha)\delta + \delta^2\varphi(\alpha, \delta) = 0$$

for some smooth function φ. We have

$$F(0, 0) = 0, \quad \left.\frac{\partial F}{\partial \delta}\right|_{(0,0)} = -2f_2(0) \neq 0, \quad \left.\frac{\partial F}{\partial \alpha}\right|_{(0,0)} = g'(0),$$

which implies (local) existence and uniqueness of a smooth function $\delta = \delta(\alpha)$ such that $\delta(0) = 0$ and $F(\alpha, \delta(\alpha)) \equiv 0$. It follows that

$$\delta(\alpha) = \frac{g'(0)}{2f_2(0)}\alpha + O(\alpha^2).$$

The map written in terms of ξ is given by

$$\tilde{\xi} \ = \ \left[f_0'(0)\alpha + \alpha^2\psi(\alpha)\right] \ + \ \xi \ + \ [f_2(0) + O(\alpha)]\,\xi^2 \ + \ O(\xi^3), \qquad (4.5)$$

where ψ is some smooth function.

Consider as a new parameter $\mu = \mu(\alpha)$ the constant (ξ-independent) term of (4.5):

$$\mu = f_0'(0)\alpha + \alpha^2\psi(\alpha).$$

We have

(a) $\mu(0) = 0$;
(b) $\mu'(0) = f_0'(0) = f_\alpha(0, 0)$.

If we assume

(A.2) $$f_\alpha(0, 0) \neq 0,$$

then the Inverse Function Theorem implies local existence and uniqueness of a smooth inverse function $\alpha = \alpha(\mu)$ with $\alpha(0) = 0$. Therefore, equation (4.5) now reads

$$\tilde{\xi} = \mu + \xi + a(\mu)\xi^2 + O(\xi^3),$$

of a smooth inverse function $\alpha = \alpha(\mu)$ with $\alpha(0) = 0$. Therefore, equation (4.5) now reads

$$\tilde{\xi} = \mu + \xi + a(\mu)\xi^2 + O(\xi^3),$$

where $a(\mu)$ is a smooth function with $a(0) = f_2(0) \neq 0$ due to the first assumption (A.1).

Let $\eta = |a(\mu)|\xi$ and $\beta = |a(\mu)|\mu$. Then we get

$$\tilde{\eta} = \beta + \eta + s\eta^2 + O(\eta^3),$$

where $s = \text{sign } a(0) = \pm 1$. □

Using Lemma 4.1, we can also eliminate $O(\eta^3)$ terms and finally arrive at the following general result.

Theorem 4.2 (Topological normal form for the fold bifurcation) *Any generic scalar one-parameter system*

$$x \mapsto f(x, \alpha),$$

having at $\alpha = 0$ the fixed point $x_0 = 0$ with $\mu = f_x(0,0) = 1$, is locally topologically equivalent near the origin to one of the following normal forms:

$$\eta \mapsto \beta + \eta \pm \eta^2. \;\square$$

Remark:

The genericity conditions in Theorem 4.2 are the nondegeneracy condition (A.1) and the transversality condition (A.2) from Theorem 4.1. ♢

4.4 The normal form of the flip bifurcation

Consider the following one-dimensional dynamical system depending on one parameter:

$$x \mapsto -(1+\alpha)x + x^3 \equiv f(x,\alpha) \equiv f_\alpha(x). \tag{4.6}$$

The map f_α is invertible for small $|\alpha|$ in a neighborhood of the origin. System (4.6) has the fixed point $x_0 = 0$ for all α with multiplier $\mu = -(1+\alpha)$. The point is linearly stable for small $\alpha < 0$ and is linearly unstable for $\alpha > 0$. At $\alpha = 0$ the point is not hyperbolic, since the multiplier $\mu = f_x(0,0) = -1$, but is nevertheless (nonlinearly) stable. There are no other fixed points near the origin for small $|\alpha|$.

Consider now the *second iterate* $f_\alpha^2(x)$ of the map (4.6). If $y = f_\alpha(x)$, then

$$\begin{aligned} f_\alpha^2(x) &= f_\alpha(y) = -(1+\alpha)y + y^3 \\ &= -(1+\alpha)[-(1+\alpha)x + x^3] + [-(1+\alpha)x + x^3]^3 \\ &= (1+\alpha)^2 x - [(1+\alpha)(2+2\alpha+\alpha^2)]x^3 + O(x^5). \end{aligned}$$

The map f_α^2 obviously has the trivial fixed point $x_0 = 0$. It also has *two* nontrivial fixed points for small $\alpha > 0$:

$$x_{1,2} = f_\alpha^2(x_{1,2}),$$

nontrivial fixed points for small $\alpha > 0$:

$$x_{1,2} = f_\alpha^2(x_{1,2}),$$

where $x_{1,2} = \pm(\sqrt{\alpha} + O(\alpha))$ (see Figure 4.4). These two points are stable

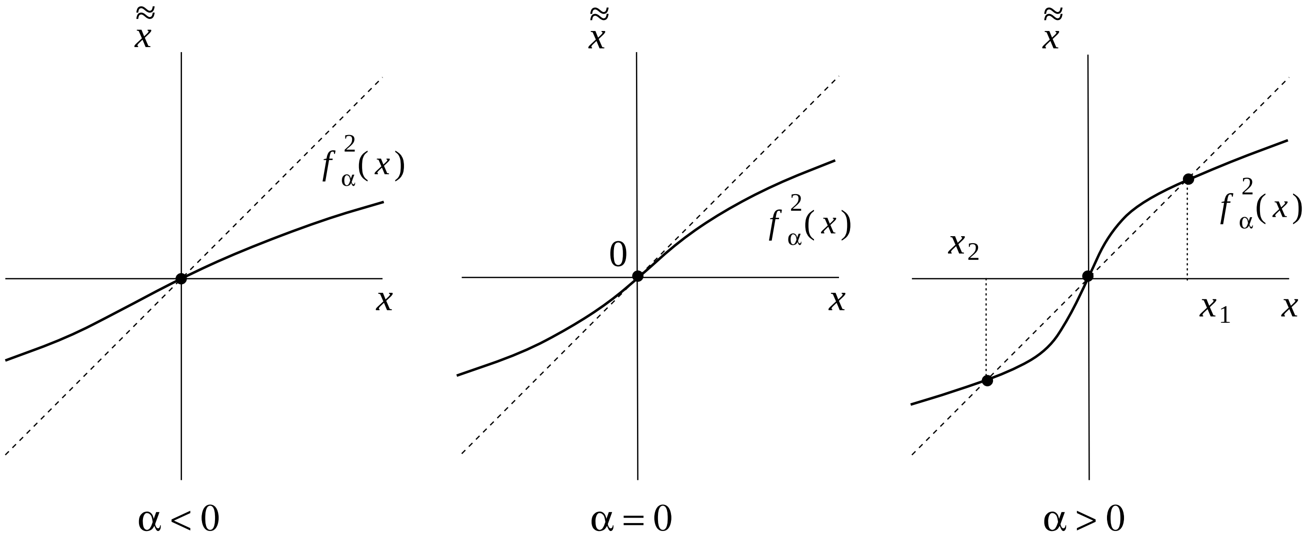


FIGURE 4.4. Second iterate map near a flip bifurcation.

and constitute a *cycle of period two* for the original map f_α. This means that

$$x_2 = f_\alpha(x_1),\ x_1 = f_\alpha(x_2),$$

with $x_1 \neq x_2$. Figure 4.5 shows the complete bifurcation diagram of system

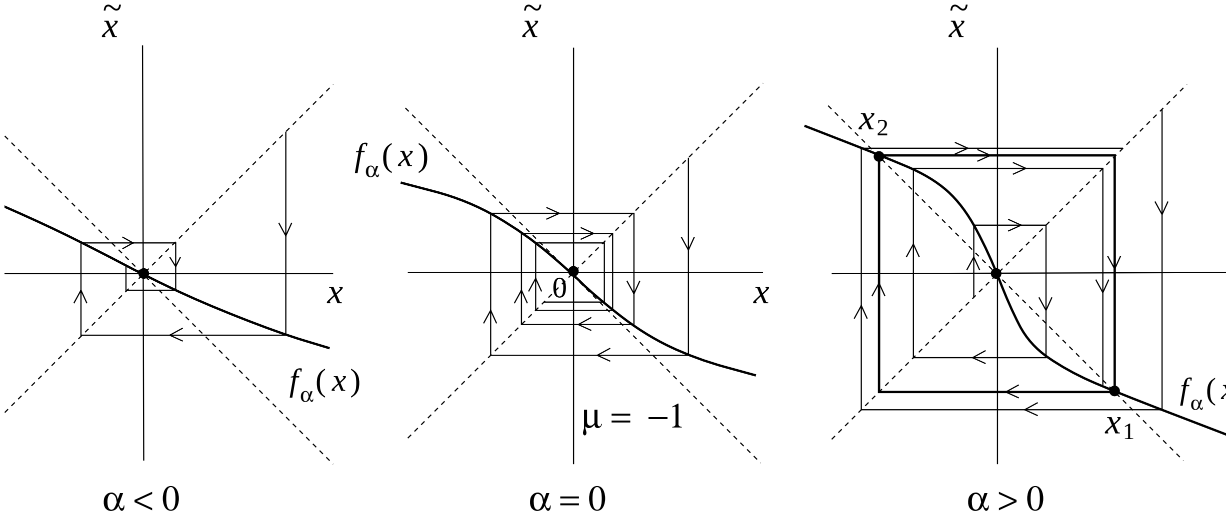
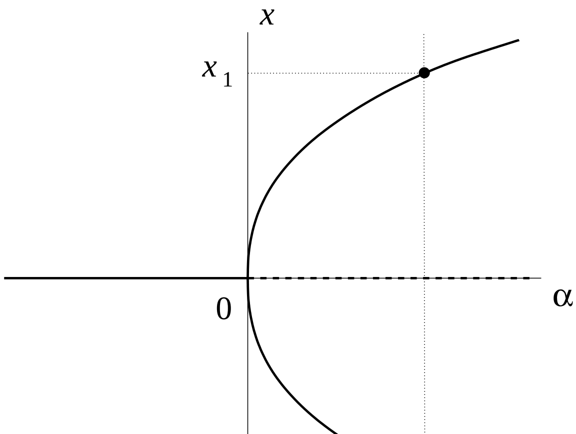


FIGURE 4.5. Flip bifurcation.



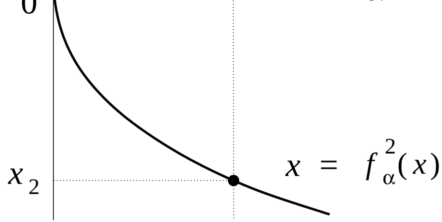


FIGURE 4.6. A flip corresponds to a pitchfork bifurcation of the second iterate.

(4.6) with the help of a staircase diagram. As α approaches zero from above, the period-two cycle "shrinks" and disappears. This is a flip bifurcation.

The other way to present this bifurcation is to use the (x, α)-plane (see Figure 4.6). In this figure, the horizontal axis corresponds to the fixed point of (4.6) (stable for $\alpha < 0$ and unstable for $\alpha > 0$), while the "parabola" represents the stable cycle of period two $\{x_1, x_2\}$ existing for $\alpha > 0$.

As usual, let us consider the effect of higher-order terms on system (4.6).

Lemma 4.2 *The system*

$$x \mapsto -(1+\alpha)x + x^3 + O(x^4)$$

is locally topologically equivalent near the origin to the system

$$x \mapsto -(1+\alpha)x + x^3. \quad \square$$

The analysis of the fixed point and the period-two cycle is a simple exercise. The rest of the proof is not easy and is omitted here.

The case

$$x \mapsto -(1+\alpha)x - x^3 \tag{4.7}$$

can be treated in the same way. For $\alpha \neq 0$, the fixed point $x_0 = 0$ has the same stability as in (4.6). At the critical parameter value $\alpha = 0$ the fixed point is unstable. The analysis of the second iterate of (4.7) reveals an *unstable* cycle of period two for $\alpha < 0$ which disappears at $\alpha = 0$. The higher-order terms do not affect the bifurcation diagram.

Remark:

By analogy with the Andronov-Hopf bifurcation, the flip bifurcation in system (4.6) is called *supercritical* or *"soft,"* while the flip bifurcation in system (4.7) is referred to as *subcritical* or *"sharp."* The bifurcation type is determined by the stability of the fixed point at the critical parameter value. ◊

4.5 Generic flip bifurcation

Theorem 4.3 *Suppose that a one-dimensional system*

$$x \mapsto f(x,\alpha), \quad x \in \mathbb{R}^1, \ \alpha \in \mathbb{R}^1,$$

with smooth f, has at $\alpha = 0$ the fixed point $x_0 = 0$, and let $\mu = f_x(0,0) =$ -1. Assume that the following nondegeneracy conditions are satisfied:

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1,$$

with smooth f, has at $\alpha = 0$ the fixed point $x_0 = 0$, and let $\mu = f_x(0,0) = -1$. Assume that the following nondegeneracy conditions are satisfied:

(B.1) $\frac{1}{2}(f_{xx}(0,0))^2 + \frac{1}{3}f_{xxx}(0,0) \neq 0$;
(B.2) $f_{x\alpha}(0,0) \neq 0$.

Then there are smooth invertible coordinate and parameter changes transforming the system into

$$\eta \mapsto -(1+\beta)\eta \pm \eta^3 + O(\eta^4).$$

Proof:

By the Implicit Function Theorem, the system has a unique fixed point $x_0(\alpha)$ in some neighborhood of the origin for all sufficiently small $|\alpha|$, since $f_x(0,0) \neq 1$. We can perform a coordinate shift, placing this fixed point at the origin. Therefore, we can assume without loss of generality that $x = 0$ is the fixed point of the system for $|\alpha|$ sufficiently small. Thus, the map f can be written as follows:

$$f(x, \alpha) = f_1(\alpha)x + f_2(\alpha)x^2 + f_3(\alpha)x^3 + O(x^4), \tag{4.8}$$

where $f_1(\alpha) = -[1 + g(\alpha)]$ for some smooth function g. Since $g(0) = 0$ and

$$g'(0) = f_{x\alpha}(0,0) \neq 0,$$

according to assumption (B.2), the function g is locally invertible and can be used to introduce a new parameter:

$$\beta = g(\alpha).$$

Our map (4.8) now takes the form

$$\tilde{x} = \mu(\beta)x + a(\beta)x^2 + b(\beta)x^3 + O(x^4),$$

where $\mu(\beta) = -(1+\beta)$, and the functions $a(\beta)$ and $b(\beta)$ are smooth. We have

$$a(0) = f_2(0) = \frac{1}{2} f_{xx}(0,0), \quad b(0) = \frac{1}{6} f_{xxx}(0,0).$$

Let us perform a smooth change of coordinate:

$$x = y + \delta y^2, \tag{4.9}$$

where $\delta = \delta(\beta)$ is a smooth function to be defined. The transformation (4.9) is invertible in some neighborhood of the origin, and its inverse can be found by the method of unknown coefficients:

$$y = x - \delta x^2 + 2\delta^2 x^3 + O(x^4). \tag{4.10}$$

Using (4.9) and (4.10), we get

$$\tilde{y} = \mu y + (a + \delta\mu - \delta\mu^2)y^2 + (b + 2\delta a - 2\delta\mu(\delta\mu + a) + 2\delta^2\mu^3)y^3 + O(y^4).$$

Thus, the quadratic term can be "killed" for all sufficiently small $|\beta|$ by

$$\tilde{y} = \mu y + (a + \delta\mu - \delta\mu^2)y^2 + (b + 2\delta a - 2\delta\mu(\delta\mu + a) + 2\delta^2\mu^3)y^3 + O(y^4).$$

Thus, the quadratic term can be "killed" for all sufficiently small $|\beta|$ by setting

$$\delta(\beta) = \frac{a(\beta)}{\mu^2(\beta) - \mu(\beta)}.$$

This can be done since $\mu^2(0) - \mu(0) = 2 \neq 0$, giving

$$\tilde{y} = \mu y + \left(b + \frac{2a^2}{\mu^2 - \mu}\right) y^3 + O(y^4) = -(1+\beta)y + c(\beta)y^3 + O(y^4)$$

for some smooth function $c(\beta)$, such that

$$c(0) = a^2(0) + b(0) = \frac{1}{4}(f_{xx}(0,0))^2 + \frac{1}{6}f_{xxx}(0,0). \tag{4.11}$$

Notice that $c(0) \neq 0$ by assumption (B.1).

Apply the rescaling

$$y = \frac{\eta}{\sqrt{|c(\beta)|}}.$$

In the new coordinate η the system takes the desired form:

$$\tilde{\eta} = -(1+\beta)\eta + s\eta^3 + O(\eta^4),$$

where $s = \text{sign } c(0) = \pm 1$. □

Using Lemma 4.2, we arrive at the following general result.

Theorem 4.4 (Topological normal form for the flip bifurcation) *Any generic, scalar, one-parameter system*

$$x \mapsto f(x, \alpha),$$

having at $\alpha = 0$ *the fixed point* $x_0 = 0$ *with* $\mu = f_x(0,0) = -1$, *is locally topologically equivalent near the origin to one of the following normal forms:*

$$\eta \mapsto -(1+\beta)\eta \pm \eta^3. \quad \square$$

Remark:

Of course, the genericity conditions in Theorem 4.4 are the nondegeneracy condition (B.1) and the transversality condition (B.2) from Theorem 4.3. ♢

Example 4.1 (Ricker's equation) Consider the following simple population model [Ricker 1954]:

$$x_{k+1} = \alpha x_k e^{-x_k},$$

where x_k is the population density in year k, and $\alpha > 0$ is the growth rate. The function on the right-hand side takes into account the negative role of interpopulation competition at high population densities. The above recurrence relation corresponds to the discrete-time dynamical system role of interpopulation competition at high population densities. The above recurrence relation corresponds to the discrete-time dynamical system

$$x \mapsto \alpha x e^{-x} \equiv f(x, \alpha). \tag{4.12}$$

System (4.12) has a trivial fixed point $x_0 = 0$ for all values of the parameter α. At $\alpha_0 = 1$, however, a nontrivial positive fixed point appears:

$$x_1(\alpha) = \ln \alpha.$$

The multiplier of this point is given by the expression

$$\mu(\alpha) = 1 - \ln \alpha.$$

Thus, x_1 is stable for $1 < \alpha < \alpha_1$ and unstable for $\alpha > \alpha_1$, where $\alpha_1 = e^2 = 7.38907\ldots$. At the critical parameter value $\alpha = \alpha_1$, the fixed point has multiplier $\mu(\alpha_1) = -1$. Therefore, a flip bifurcation takes place. To apply Theorem 4.4, we need to verify the corresponding nondegeneracy conditions in which all the derivatives must be computed at the fixed point $x_1(\alpha_1) = 2$ and at the critical parameter value α_1.

One can check that

$$c(0) = \frac{1}{6} > 0, \quad f_{x\alpha} = -\frac{1}{e^2} \neq 0.$$

Therefore, a unique and stable period-two cycle bifurcates from x_1 for $\alpha > \alpha_1$.

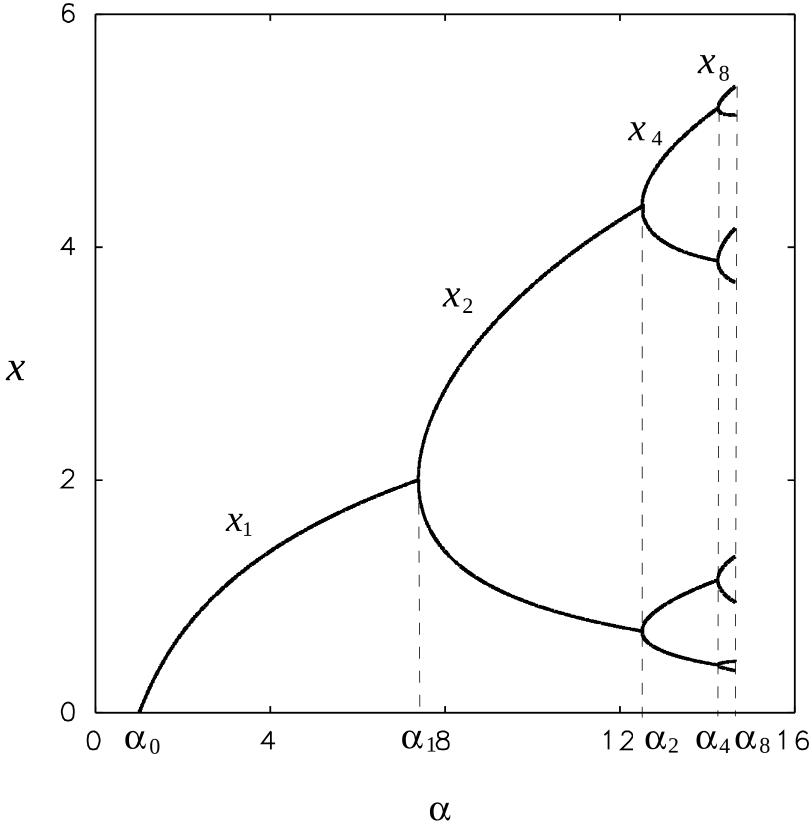


FIGURE 4.7. Cascade of period-doubling (flip) bifurcations in Ricker's equation.

The fate of this period-two cycle can be traced further. It can be ver-

The fate of this period-two cycle can be traced further. It can be verified numerically (see Exercise 4) that this cycle loses stability at $\alpha_2 = 12.50925\ldots$ via the flip bifurcation, giving rise to a stable period-four cycle. It bifurcates again at $\alpha_4 = 14.24425\ldots$, generating a stable period-eight cycle that loses its stability at $\alpha_8 = 14.65267\ldots$. The next period doubling takes place at $\alpha_{16} = 14.74212\ldots$ (see Figure 4.7, where several doublings are presented).

It is natural to assume that there is an *infinite* sequence of bifurcation values: $\alpha_{m(k)},\ m(k) = 2^k,\ k = 1, 2, \ldots$ ($m(k)$ is the period of the cycle before the kth doubling). Moreover, one can check that at least the first few elements of this sequence closely resemble a *geometric progression*. In fact, the quotient

$$\frac{\alpha_{m(k)} - \alpha_{m(k-1)}}{\alpha_{m(k+1)} - \alpha_{m(k)}}$$

tends to $\mu_F = 4.6692\ldots$ as k increases. This phenomenon is called *Feigenbaum's cascade* of period doublings, and the constant μ_F is referred to as the *Feigenbaum constant*. The most surprising fact is that this constant is the same for many different systems exhibiting a cascade of flip bifurcations. This universality has a deep reasoning, which will be discussed in Appendix 1 to this chapter. $\diamond$

4.6 The “normal form” of the Neimark-Sacker bifurcation

Consider the following two-dimensional discrete-time system depending on one parameter:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (1+\alpha)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2+x_2^2)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{4.13}$$

where α is the parameter; $\theta = \theta(\alpha), a = a(\alpha)$, and $b = b(\alpha)$ are smooth functions; and $0 < \theta(0) < \pi,\ a(0) \neq 0$.

This system has the fixed point $x_1 = x_2 = 0$ for all α with Jacobian matrix

$$A = (1+\alpha)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The matrix has eigenvalues $\mu_{1,2} = (1+\alpha)e^{\pm i\theta}$, which makes the map (4.13) invertible near the origin for all small $|\alpha|$. As can be seen, the fixed point at the origin is nonhyperbolic at $\alpha = 0$ due to a complex-conjugate pair of the eigenvalues on the unit circle. To analyze the corresponding bifurcation, introduce the complex variable $z = x_1 + ix_2, \bar{z} = x_1 - ix_2, |z|^2 = z\bar{z} = x_1^2 + x_2^2$, and set $d = a + ib$. The equation for z reads eigenvalues on the unit circle. To analyze the corresponding bifurcation, introduce the complex variable $z = x_1 + ix_2, \bar{z} = x_1 - ix_2, |z|^2 = z\bar{z} = x_1^2 + x_2^2$, and set $d = a + ib$. The equation for z reads

$$z \mapsto e^{i\theta} z(1 + \alpha + d|z|^2) = \mu z + cz|z|^2,$$

where $\mu = \mu(\alpha) = (1+\alpha)e^{i\theta(\alpha)}$ and $c = c(\alpha) = e^{i\theta(\alpha)}d(\alpha)$ are complex functions of the parameter α.

Using the representation $z = \rho e^{i\varphi}$, we obtain for $\rho = |z|$

$$\rho \mapsto \rho|1 + \alpha + d(\alpha)\rho^2|.$$

Since

$$\begin{aligned} |1 + \alpha + d(\alpha)\rho^2| &= (1+\alpha)\left(1 + \frac{2a(\alpha)}{1+\alpha}\rho^2 + \frac{|d(\alpha)|^2}{(1+\alpha)^2}\rho^4\right)^{1/2} \\ &= 1 + \alpha + a(\alpha)\rho^2 + O(\rho^3), \end{aligned}$$

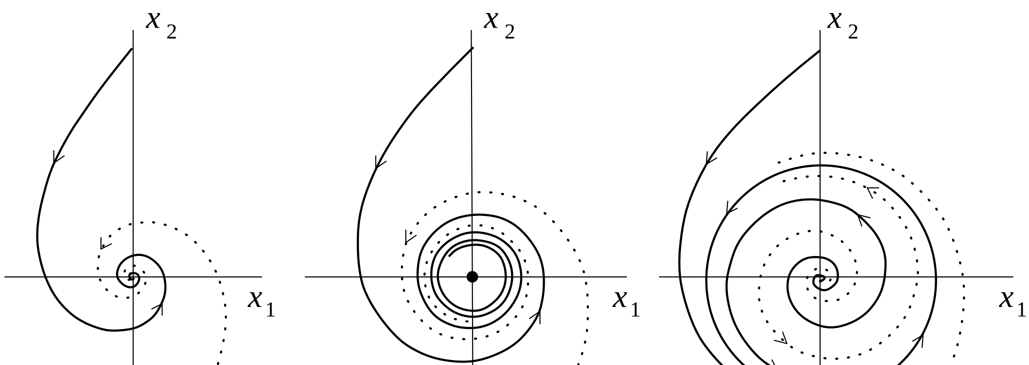
we obtain the following *polar* form of system (4.13):

$$\begin{cases} \rho & \mapsto \quad \rho(1 + \alpha + a(\alpha)\rho^2) + \rho^4 R_\alpha(\rho), \\ \varphi & \mapsto \quad \varphi + \theta(\alpha) + \rho^2 Q_\alpha(\rho), \end{cases} \tag{4.14}$$

for functions R and Q, which are smooth functions of (ρ, α). Bifurcations of the systems's phase portrait as α passes through zero can easily be analyzed using the latter form, since the mapping for ρ is *independent* of φ. The first equation in (4.14) defines a one-dimensional dynamical system that has the fixed point $\rho = 0$ for all values of α. The point is linearly stable if $\alpha < 0$; for $\alpha > 0$ the point becomes linearly unstable. The stability of the fixed point at $\alpha = 0$ is determined by the sign of the coefficient $a(0)$. Suppose that $a(0) < 0$; then the origin is (nonlinearly) stable at $\alpha = 0$. Moreover, the ρ-map of (4.14) has an additional stable fixed point

$$\rho_0(\alpha) = \sqrt{-\frac{\alpha}{a(\alpha)}} + O(\alpha)$$

for $\alpha > 0$. The φ-map of (4.14) describes a rotation by an angle depending on ρ and α; it is approximately equal to $\theta(\alpha)$. Thus, by superposition of the mappings defined by (4.14), we obtain the bifurcation diagram for the original two-dimensional system (4.13) (see Figure 4.8).



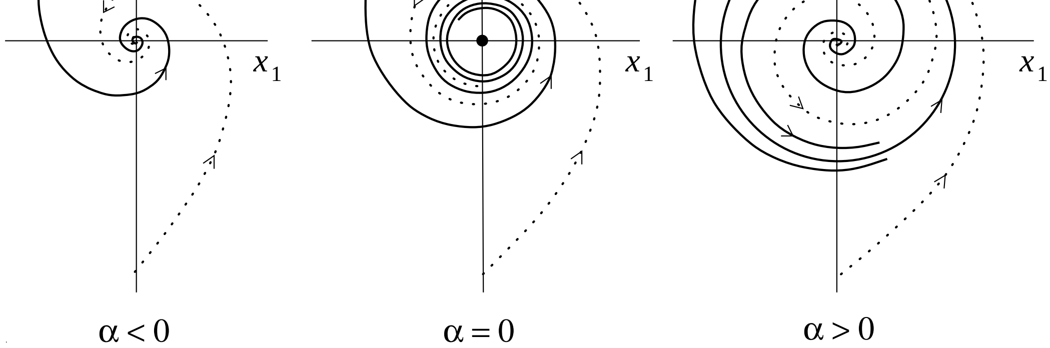


FIGURE 4.8. Supercritical Neimark-Sacker bifurcation.

The system always has a fixed point at the origin. This point is stable for $\alpha < 0$ and unstable for $\alpha > 0$. The invariant curves of the system near the origin look like the orbits near the stable focus of a continuous-time system for $\alpha < 0$ and like orbits near the unstable focus for $\alpha > 0$. At the critical parameter value $\alpha = 0$ the point is nonlinearly stable. The fixed point is surrounded for $\alpha > 0$ by an isolated *closed invariant curve* that is unique and stable. The curve is a circle of radius $\rho_0(\alpha)$. All orbits starting outside or inside the closed invariant curve, except at the origin, tend to the curve under iterations of (4.14). This is a Neimark-Sacker bifurcation.

This bifurcation can also be presented in (x_1, x_2, α)-space. The appearing family of closed invariant curves, parametrized by α, forms a *paraboloid* surface.

The case $a(0) > 0$ can be analyzed in the same way. The system undergoes the Neimark-Sacker bifurcation at $\alpha = 0$. Contrary to the considered

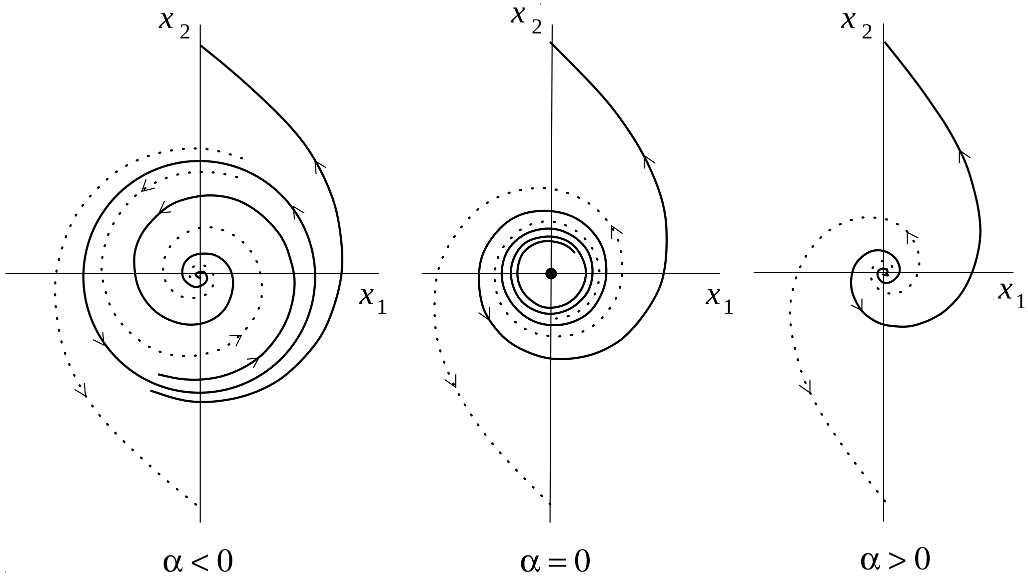


FIGURE 4.9. Subcritical Neimark-Sacker bifurcation.

case, there is an *unstable* closed invariant curve that disappears when α crosses zero from negative to positive values (see Figure 4.9).

Remarks:

(1) As in the cases of the Andronov-Hopf and the flip bifurcations, these two cases are often called *supercritical* and *subcritical* (or, better, *"soft"* and *"sharp"*) Neimark-Sacker bifurcations. As usual, the type of the bifurcation is determined by the stability of the fixed point at the bifurcation parameter

two cases are often called *supercritical* and *subcritical* (or, better, "*soft*" and "*sharp*") Neimark-Sacker bifurcations. As usual, the type of the bifurcation is determined by the stability of the fixed point at the bifurcation parameter value.

(2) The structure of orbits of (4.14) on the invariant circle depends on whether the ratio between the rotation angle $\Delta\varphi = \theta(\alpha) + \rho^2 Q_\alpha(\rho)$ and 2π is rational or irrational on the circle. If it is rational, all the orbits on the curve are *periodic*. More precisely, if

$$\frac{\Delta\varphi}{2\pi} = \frac{p}{q}$$

with integers p and q, all the points on the curve are cycles of period q of the pth iterate of the map. If the ratio is irrational, there are no periodic orbits and all the orbits are dense in the circle. $\diamondsuit$

Let us now add higher-order terms to system (4.13); for instance, consider the system

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\mapsto (1+\alpha)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\quad + (x_1^2 + x_2^2)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + O(\|x\|^4).\end{aligned} \tag{4.15}$$

Here, the $O(\|x\|^4)$ terms can depend smoothly on α. Unfortunately, it cannot be said that system (4.15) is locally topologically equivalent to system

(4.13). In this case, the higher-order terms do affect the bifurcation behavior of the system. If one writes (4.15) in the polar form, the mapping for ρ will depend on φ. The system can be represented in a form similar to (4.14) but with 2π-periodic functions R and Q. Nevertheless, the phase portraits of systems (4.13) and (4.15) have some important features in common. Namely, the following lemma holds.

Lemma 4.3 *$O(\|x\|^4)$ terms do not affect the bifurcation of the closed invariant curve in* (4.15). *That is, a locally unique invariant curve bifurcates from the origin in the same direction and with the same stability as in system* (4.13). □

The proof of the lemma is rather involved and is given in Appendix 2. The geometrical idea behind the proof is simple. We expect that map (4.15) has an invariant curve near the invariant circle of the map (4.13). Fix α and consider the circle

$$S_0 = \left\{ (\rho, \varphi) : \rho = \sqrt{-\frac{\alpha}{a(\alpha)}} \right\},$$

which is located near the invariant circle of the "unperturbed" map without $O(\|x\|^4)$ terms. It can be shown that iterations $F^k S_0, k = 1, 2, \ldots$, where F is the map defined by (4.15), converge to a closed invariant curve

$$S_\infty = \{(\rho, \varphi) : \rho = \Psi(\varphi)\},$$

F is the map defined by (4.15), converge to a closed invariant curve

$$S_\infty = \{(\rho, \varphi) : \rho = \Psi(\varphi)\},$$

which is not a circle but is close to S_0. Here, Ψ is a 2π-periodic function of φ describing S_∞ in polar coordinates. To establish the convergence, we have to introduce a new "radial" variable u in a band around S_0 (both the band diameter and its width "shrink" as $\alpha \to 0$) and show that the map F defines a *contraction* map $\mathcal{F}$ on a proper function space of 2π-periodic functions $u = u(\varphi)$. Then the Contraction Mapping Principle (see Chapter 1) gives the existence of a fixed point $u^{(\infty)}$ of $\mathcal{F} : \mathcal{F}(u^{(\infty)}) = u^{(\infty)}$. The periodic function $u^{(\infty)}(\varphi)$ represents the closed invariant curve S_∞ we are looking for at α fixed. Uniqueness and stability of S_∞ in the band follow, essentially, from the contraction. It can be verified that outside the band there are no nontrivial invariant sets of (4.15).

Remarks:

(1) The orbit structure on the closed invariant curve and the variation of this structure when the parameter changes are generically different in systems (4.13) and (4.15). We will return to the analysis of bifurcations on the invariant curve in Chapter 7. Here we just notice that, generically, there is only a *finite* number of periodic orbits on the closed invariant curve. Let $a(0) < 0$. Then, some iterate p of map (4.15) can have two q-periodic orbits: a totally stable "node" cycle of period q and a saddle cycle of period q (see Figure 4.10). The cycles exist in some "parameter window" and disappear

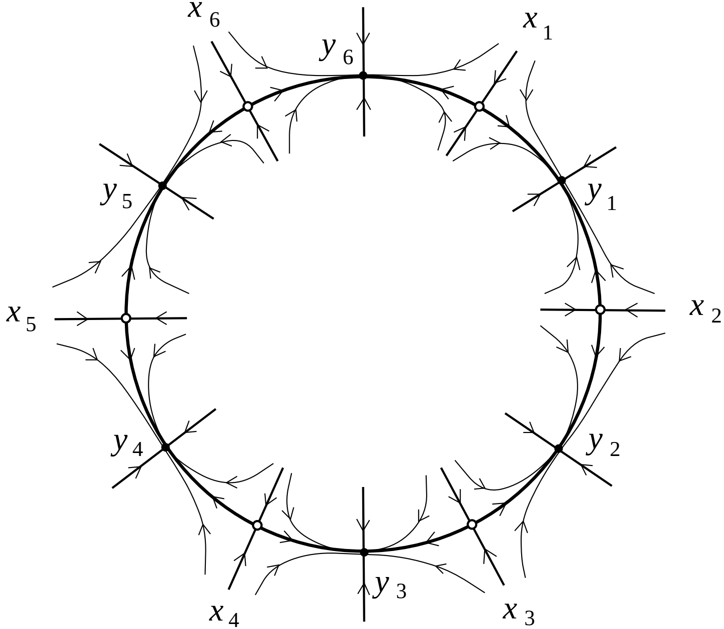


FIGURE 4.10. Saddle $\{x_1, x_2, \ldots, x_6\}$ and stable $\{y_1, y_2, \ldots, y_6\}$ period-six orbits on the invariant circle.

on its borders through the fold bifurcation. A generic system exhibits an infinite number of such bifurcations corresponding to different windows.

(2) The bifurcating invariant closed curve in (4.15) has *finite* smoothness: The function $\Psi(\varphi)$ representing it in polar coordinates generically has only a finite number of continuous derivatives with respect to φ, even if the map (4.15) is differentiable infinitely many times. The number increases as

The function $\Psi(\varphi)$ representing it in polar coordinates generically has only a finite number of continuous derivatives with respect to φ, even if the map (4.15) is differentiable infinitely many times. The number increases as $|\alpha| \to 0$. The nonsmoothness appears when the saddle's unstable (stable) manifolds meet at the "node" points. $\diamondsuit$

4.7 Generic Neimark-Sacker bifurcation

We now shall prove that any generic two-dimensional system undergoing a Neimark-Sacker bifurcation can be transformed into the form (4.15).

Consider a system

$$x \mapsto f(x, \alpha), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1,$$

with a smooth function f, which has at $\alpha = 0$ the fixed point $x = 0$ with simple eigenvalues $\mu_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$. By the Implicit Function Theorem, the system has a unique fixed point $x_0(\alpha)$ in some neighborhood of the origin for all sufficiently small $|\alpha|$, since $\mu = 1$ is not an eigenvalue of the Jacobian matrix.[1] We can perform a parameter-dependent coordinate shift, placing this fixed point at the origin. Therefore, we may assume

[1] Since $\mu = 0$ is not an eigenvalue, the system is invertible in some neighborhood of the origin for sufficiently small $|\alpha|$.

without loss of generality that $x = 0$ is the fixed point of the system for $|\alpha|$ sufficiently small. Thus, the system can be written as

$$x \mapsto A(\alpha)x + F(x, \alpha), \tag{4.16}$$

where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in x starting with at least quadratic terms, $F(0, \alpha) = 0$ for all sufficiently small $|\alpha|$. The Jacobian matrix $A(\alpha)$ has two multipliers

$$\mu_{1,2}(\alpha) = r(\alpha)e^{\pm i\varphi(\alpha)},$$

where $r(0) = 1, \varphi(0) = \theta_0$. Thus, $r(\alpha) = 1+\beta(\alpha)$ for some smooth function $\beta(\alpha), \beta(0) = 0$. Suppose that $\beta'(0) \neq 0$. Then, we can use β as a new parameter and express the multipliers in terms of $\beta : \mu_1(\beta) = \mu(\beta), \mu_2(\beta) = \bar{\mu}(\beta)$, where

$$\mu(\beta) = (1 + \beta)e^{i\theta(\beta)}$$

with a smooth function $\theta(\beta)$ such that $\theta(0) = \theta_0$.

Lemma 4.4 *By the introduction of a complex variable and a new parameter, system* (4.16) *can be transformed for all sufficiently small* $|\alpha|$ *into the following form:*

$$z \mapsto \mu(\beta)z + g(z, \bar{z}, \beta), \tag{4.17}$$

where $\beta \in \mathbb{R}^1, z \in \mathbb{C}^1, \mu(\beta) = (1 + \beta)e^{i\theta(\beta)}$, *and* g *is a complex-valued smooth function of* $z, \bar{z}$, *and* β *whose Taylor expansion with respect to* $(z, \bar{z})$

where $\beta \in \mathbb{R}^1, z \in \mathbb{C}^1, \mu(\beta) = (1+\beta)e^{i\theta(\beta)}$, *and* g *is a complex-valued smooth function of* $z, \bar{z}$, *and* β *whose Taylor expansion with respect to* $(z, \bar{z})$ *contains quadratic and higher-order terms*:

$$g(z, \bar{z}, \beta) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(\beta) z^k \bar{z}^l,$$

with $k, l = 0, 1, \ldots$. □

The proof of the lemma is completely analogous to that from the Andronov-Hopf bifurcation analysis in Chapter 3 and is left as an exercise for the reader.

As in the Andronov-Hopf case, we start by making *nonlinear* (complex) coordinate changes that will simplify the map (4.17). First, we remove all the quadratic terms.

Lemma 4.5 *The map*

$$z \mapsto \mu z + \frac{g_{20}}{2} z^2 + g_{11} z\bar{z} + \frac{g_{02}}{2} \bar{z}^2 + O(|z|^3), \tag{4.18}$$

where $\mu = \mu(\beta) = (1+\beta)e^{i\theta(\beta)}, g_{ij} = g_{ij}(\beta)$, *can be transformed by an invertible parameter-dependent change of complex coordinate*

$$z = w + \frac{h_{20}}{2} w^2 + h_{11} w\bar{w} + \frac{h_{02}}{2} \bar{w}^2,$$

for all sufficiently small $|\beta|$, *into a map without quadratic terms*:

$$w \mapsto \mu w + O(|w|^3),$$

provided that

$$e^{i\theta_0} \neq 1 \quad \textit{and} \quad e^{3i\theta_0} \neq 1.$$

Proof:

The inverse change of variables is given by

$$w = z - \frac{h_{20}}{2}z^2 - h_{11}z\bar{z} - \frac{h_{02}}{2}\bar{z}^2 + O(|z|^3).$$

Therefore, in the new coordinate w, the map (4.18) takes the form

$$\begin{aligned}
\tilde{w} = \mu w \quad &+ \quad \frac{1}{2}(g_{20} + (\mu - \mu^2)h_{20})w^2 \\
&+ \quad (g_{11} + (\mu - |\mu|^2)h_{11})w\bar{w} \\
&+ \quad \frac{1}{2}(g_{02} + (\mu - \bar{\mu}^2)h_{02})\bar{w}^2 \\
&+ \quad O(|w|^3).
\end{aligned}$$

Thus, by setting

$$h_{20} = \frac{g_{20}}{\mu^2 - \mu}, \quad h_{11} = \frac{g_{11}}{|\mu|^2 - \mu}, \quad h_{02} = \frac{g_{02}}{\bar{\mu}^2 - \mu},$$

we "kill" all the quadratic terms in (4.18). These substitutions are valid if the denominators are nonzero for all sufficiently small $|\beta|$ including $\beta = 0$.

$$\mu^2 - \mu \qquad |\mu|^2 - \mu \qquad \mu^2 - \mu$$

we "kill" all the quadratic terms in (4.18). These substitutions are valid if the denominators are nonzero for all sufficiently small $|\beta|$ including $\beta = 0$. Indeed, this is the case, since

$$\begin{aligned}
\mu^2(0) - \mu(0) &= e^{i\theta_0}(e^{i\theta_0} - 1) \neq 0, \\
|\mu(0)|^2 - \mu(0) &= 1 - e^{i\theta_0} \neq 0, \\
\bar{\mu}(0)^2 - \mu(0) &= e^{i\theta_0}(e^{-3i\theta_0} - 1) \neq 0,
\end{aligned}$$

due to our restrictions on θ_0. $\Box$

Remarks:

(1) Let $\mu_0 = \mu(0)$. Then, the conditions on θ_0 used in the lemma can be written as

$$\mu_0 \neq 1, \;\; \mu_0^3 \neq 1.$$

Notice that the first condition holds automatically due to our initial assumptions on θ_0.

(2) The resulting coordinate transformation is polynomial with coefficients that are smoothly dependent on β. In some neighborhood of the origin the transformation is *near-identical.*

(3) Notice the transformation *changes* the coefficients of the cubic terms of (4.18). $\diamondsuit$

Assuming that we have removed all quadratic terms, let us try to eliminate the cubic terms as well.

Lemma 4.6 *The map*

$$z \mapsto \mu z + \frac{g_{30}}{6} z^3 + \frac{g_{21}}{2} z^2 \bar{z} + \frac{g_{12}}{2} z \bar{z}^2 + \frac{g_{03}}{6} \bar{z}^3 + O(|z|^4), \tag{4.19}$$

where $\mu = \mu(\beta) = (1+\beta)e^{i\theta(\beta)}$, $g_{ij} = g_{ij}(\beta)$, *can be transformed by an invertible parameter-dependent change of coordinates*

$$z = w + \frac{h_{30}}{6} w^3 + \frac{h_{21}}{2} w^2 \bar{w} + \frac{h_{12}}{2} w \bar{w}^2 + \frac{h_{03}}{6} \bar{w}^3,$$

for all sufficiently small $|\beta|$, *into a map with only one cubic term:*

$$w \mapsto \mu w + c_1 w^2 \bar{w} + O(|w|^4),$$

provided that

$$e^{2i\theta_0} \neq 1 \quad and \quad e^{4i\theta_0} \neq 1.$$

Proof:

The inverse transformation is

$$w = z - \frac{h_{30}}{6} z^3 - \frac{h_{21}}{2} z^2 \bar{z} - \frac{h_{12}}{2} z \bar{z}^2 - \frac{h_{03}}{6} \bar{z}^3 + O(|z|^4).$$

Therefore,

$$\tilde{w} = \lambda w + \frac{1}{6}(g_{30} + (\mu - \mu^3)h_{30})w^3 + \frac{1}{2}(g_{21} + (\mu - \mu|\mu|^2)h_{21})w^2\bar{w}$$

Therefore,

$$\begin{aligned}\tilde{w} &= \lambda w + \frac{1}{6}(g_{30} + (\mu - \mu^3)h_{30})w^3 + \frac{1}{2}(g_{21} + (\mu - \mu|\mu|^2)h_{21})w^2\bar{w} \\ &+ \frac{1}{2}(g_{12} + (\mu - \bar{\mu}|\mu|^2)h_{12})w\bar{w}^2 + \frac{1}{6}(g_{03} + (\mu - \bar{\mu}^3)h_{03})\bar{w}^3 + O(|w|^4).\end{aligned}$$

Thus, by setting

$$h_{30} = \frac{g_{30}}{\mu^3 - \mu}, \quad h_{12} = \frac{g_{12}}{\bar{\mu}|\mu|^2 - \mu}, \quad h_{03} = \frac{g_{03}}{\bar{\mu}^3 - \mu},$$

we can annihilate all cubic terms in the resulting map except the $w^2\bar{w}$-term, which must be treated separately. The substitutions are valid since all the involved denominators are nonzero for all sufficiently small $|\beta|$ due to the assumptions concerning θ_0.

One can also try to eliminate the $w^2\bar{w}$-term by formally setting

$$h_{21} = \frac{g_{21}}{\mu(1 - |\mu|^2)}.$$

This is possible for small $\beta \neq 0$, but the denominator vanishes at $\beta = 0$ *for all* θ_0. Thus, no extra conditions on θ_0 would help. To obtain a transformation that is smoothly dependent on β, set $h_{21} = 0$, that results in

$$c_1 = \frac{g_{21}}{2}. \ \square$$

Remarks:

(1) The conditions imposed on θ_0 in the lemma mean

$$\mu_0^2 \neq 1, \;\; \mu_0^4 \neq 1,$$

and therefore, in particular, $\mu_0 \neq -1$ and $\mu_0 \neq i$. The first condition holds automatically due to our initial assumptions on θ_0.

(2) The remaining cubic $w^2\bar{w}$-term is called a *resonant term*. Note that its coefficient is the *same* as the coefficient of the cubic term $z^2\bar{z}$ in the original map (4.19). ♢

We now combine the two previous lemmas.

Lemma 4.7 (Normal form for the Neimark-Sacker bifurcation)
The map

$$\begin{aligned} z \mapsto \mu z &+ \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 \\ &+ \frac{g_{30}}{6}z^3 + \frac{g_{21}}{2}z^2\bar{z} + \frac{g_{12}}{2}z\bar{z}^2 + \frac{g_{03}}{6}\bar{z}^3 \\ &+ O(|z|^4), \end{aligned}$$

where $\mu = \mu(\beta) = (1+\beta)e^{i\theta(\beta)}$, $g_{ij} = g_{ij}(\beta)$, *and* $\theta_0 = \theta(0)$ *is such that* $e^{ik\theta_0} \neq 1$ *for* $k = 1, 2, 3, 4$, *can be transformed by an invertible parameter-dependent change of complex coordinate, which is smoothly dependent on the parameter,*

$e^{ik\theta_0} \neq 1$ for $k = 1, 2, 3, 4$, can be transformed by an invertible parameter-dependent change of complex coordinate, which is smoothly dependent on the parameter,

$$\begin{aligned} z = w \quad &+ \quad \frac{h_{20}}{2} w^2 + h_{11} w \bar{w} + \frac{h_{02}}{2} \bar{w}^2 \\ &+ \quad \frac{h_{30}}{6} w^3 + \frac{h_{12}}{2} w \bar{w}^2 + \frac{h_{03}}{6} \bar{w}^3, \end{aligned}$$

for all sufficiently small $|\beta|$, into a map with only the resonant cubic term:

$$w \mapsto \mu w + c_1 w^2 \bar{w} + O(|w|^4),$$

where $c_1 = c_1(\beta)$. □

The truncated superposition of the transformations defined in the two previous lemmas gives the required coordinate change. First, annihilate all the quadratic terms. This will also change the coefficients of the cubic terms. The coefficient of $w^2\bar{w}$ will be $\frac{1}{2}\tilde{g}_{21}$, say, instead of $\frac{1}{2}g_{21}$. Then, eliminate all the cubic terms except the resonant one. The coefficient of this term remains $\frac{1}{2}\tilde{g}_{21}$. Thus, all we need to compute to get the coefficient of c_1 in terms of the given equation is a new coefficient $\frac{1}{2}\tilde{g}_{21}$ of the $w^2\bar{w}$-term after the *quadratic* transformation. The computations result in the following expression for $c_1(\alpha)$:

$$c_1 = \frac{g_{20}g_{11}(\bar{\mu} - 3 + 2\mu)}{2(\mu^2 - \mu)(\bar{\mu} - 1)} + \frac{|g_{11}|^2}{1 - \bar{\mu}} + \frac{|g_{02}|^2}{2(\mu^2 - \bar{\mu})} + \frac{g_{21}}{2}, \tag{4.20}$$

which gives, for the critical value of c_1,

$$c_1(0) = \frac{g_{20}(0)g_{11}(0)(1-2\mu_0)}{2(\mu_0^2-\mu_0)} + \frac{|g_{11}(0)|^2}{1-\bar{\mu}_0} + \frac{|g_{02}(0)|^2}{2(\mu_0^2-\bar{\mu}_0)} + \frac{g_{21}(0)}{2}, \quad (4.21)$$

where $\mu_0 = e^{i\theta_0}$.

We now summarize the obtained results in the following theorem.

Theorem 4.5 *Suppose a two-dimensional discrete-time system*

$$x \mapsto f(x,\alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1,$$

with smooth f, has, for all sufficiently small $|\alpha|$, the fixed point $x = 0$ with multipliers

$$\mu_{1,2}(\alpha) = r(\alpha)e^{\pm i\varphi(\alpha)},$$

where $r(0) = 1, \varphi(0) = \theta_0$.

Let the following conditions be satisfied:

(C.1) $r'(0) \neq 0$;
(C.2) $e^{ik\theta_0} \neq 1$ *for* $k = 1, 2, 3, 4$.

Then, there are smooth invertible coordinate and parameter changes transforming the system into

Then, there are smooth invertible coordinate and parameter changes transforming the system into

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto (1+\beta) \begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} +$$
$$(y_1^2+y_2^2) \begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix} \begin{pmatrix} a(\beta) & -b(\beta) \\ b(\beta) & a(\beta) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(\|y\|^4) \tag{4.22}$$

with $\theta(0) = \theta_0$ *and* $a(0) = \mathrm{Re}(e^{-i\theta_0} c_1(0))$, *where* $c_1(0)$ *is given by the formula* (4.21)*.*

Proof:

The only thing left to verify is the formula for $a(0)$. Indeed, by Lemmas 4.4, 4.5, and 4.6, the system can be transformed to the complex Poincaré normal form,

$$w \mapsto \mu(\beta)w + c_1(\beta)w|w|^2 + O(|w|^4),$$

for $\mu(\beta) = (1+\beta)e^{i\theta(\beta)}$. This map can be written as

$$w \mapsto e^{i\theta(\beta)}(1+\beta+d(\beta)|w|^2)w + O(|w|^4),$$

where $d(\beta) = a(\beta) + ib(\beta)$ for some real functions $a(\beta)$, $b(\beta)$. A return to the real coordinates $(y_1, y_2), w = y_1 + iy_2$, gives system (4.22). Finally,

$$a(\beta) = \mathrm{Re}\ d(\beta) = \mathrm{Re}(e^{-i\theta(\beta)} c_1(\beta)).$$

Thus,

$$a(0) = \text{Re}(e^{-i\theta_0} c_1(0)). \quad \square$$

Using Lemma 4.3, we can state the following general result.

Theorem 4.6 (Generic Neimark-Sacker bifurcation) *For any generic two-dimensional one-parameter system*

$$x \mapsto f(x, \alpha),$$

having at $\alpha = 0$ the fixed point $x_0 = 0$ with complex multipliers $\mu_{1,2} = e^{\pm i\theta_0}$, there is a neighborhood of x_0 in which a unique closed invariant curve bifurcates from x_0 as α passes through zero. $\square$

Remark:

The genericity conditions assumed in the theorem are the transversality condition (C.1) and the nondegeneracy condition (C.2) from Theorem 4.5 *and* the additional nondegeneracy condition

(C.3) $a(0) \neq 0$.

It should be stressed that the conditions $e^{ik\theta_0} \neq 1$ for $k = 1, 2, 3, 4$ are not merely technical. If they are not satisfied, the closed invariant curve may not appear at all, or there might be several invariant curves bifurcating from the fixed point (see Chapter 9). $\diamondsuit$

merely technical. If they are not satisfied, the closed invariant curve may not appear at all, or there might be several invariant curves bifurcating from the fixed point (see Chapter 9). $\diamondsuit$

The coefficient $a(0)$, which determines the direction of the appearance of the invariant curve in a generic system exhibiting the Neimark-Sacker bifurcation, can be computed via

$$a(0) = \text{Re}\left(\frac{e^{-i\theta_0} g_{21}}{2}\right) - \text{Re}\left(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})} g_{20} g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2. \tag{4.23}$$

In Chapter 5 we will see how to deal with n-dimensional discrete-time systems where $n > 2$ and how to apply the results to limit cycle bifurcations in continuous-time systems.

Example 4.2 (Neimark-Sacker bifurcation in the delayed logistic equation) Consider the following recurrence equation:

$$u_{k+1} = r u_k (1 - u_{k-1}).$$

This is a simple population dynamics model, where u_k stands for the density of a population at time k, and r is the growth rate. It is assumed that the growth is determined not only by the current population density but also by its density in the past.

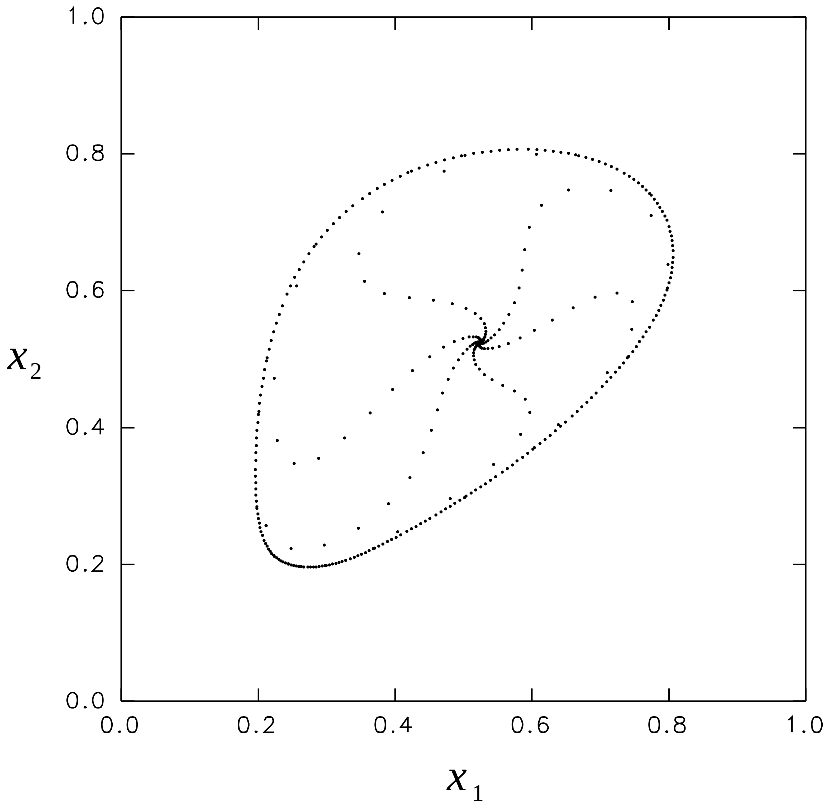


FIGURE 4.11. Stable invariant curve in the delayed logistic equation.

If we introduce $v_k = u_{k-1}$, the equation can be rewritten as

$$u_{k+1} = ru_k(1 - v_k),$$

If we introduce $v_k = u_{k-1}$, the equation can be rewritten as

$$\begin{aligned} u_{k+1} &= r u_k (1 - v_k), \\ v_{k+1} &= v_k, \end{aligned}$$

which, in turn, defines the two-dimensional discrete-time dynamical system,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} r x_1 (1 - x_2) \\ x_1 \end{pmatrix} \equiv \begin{pmatrix} F_1(x, r) \\ F_2(x, r) \end{pmatrix}, \tag{4.24}$$

where $x = (x_1, x_2)^T$. The map (4.24) has the fixed point $(0,0)^T$ for all values of r. For $r > 1$, a nontrivial positive fixed point x^0 appears, with the coordinates

$$x_1^0(r) = x_2^0(r) = 1 - \frac{1}{r}.$$

The Jacobian matrix of the map (4.24) evaluated at the nontrivial fixed point is given by

$$A(r) = \begin{pmatrix} 1 & 1 - r \\ 1 & 0 \end{pmatrix}$$

and has eigenvalues

$$\mu_{1,2}(r) = \frac{1}{2} \pm \sqrt{\frac{5}{4} - r}.$$

If $r > \frac{5}{4}$, the eigenvalues are complex and $|\mu_{1,2}|^2 = \mu_1 \mu_2 = r - 1$. Therefore, at $r = r_0 = 2$ the nontrivial fixed point loses stability and we have a

Neimark-Sacker bifurcation: The critical multipliers are

$$\mu_{1,2} = e^{\pm i\theta_0}, \ \theta_0 = \frac{\pi}{3} = 60^0.$$

It is clear that conditions (C.1) and (C.2) are satisfied.

To verify the nondegeneracy condition $(C.3)$, we have to compute $a(0)$. The critical Jacobian matrix $A_0 = A(r_0)$ have the eigenvectors

$$A_0 q = e^{i\theta_0} q, \ A_0^T p = e^{-i\theta_0} p,$$

where

$$q \sim \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ 1\right)^T, \ \ p \sim \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ 1\right)^T.$$

To achieve the normalization $\langle p, q\rangle = 1$, we can take, for example,

$$q = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ 1\right)^T, \ \ p = \left(i\frac{\sqrt{3}}{3}, \ \frac{1}{2} - i\frac{\sqrt{3}}{6}\right)^T.$$

Now we compose

$$x = x^0 + zq + \bar{z}\bar{q}$$

and evaluate the function

Now we compose

$$x = x^0 + zq + \bar{z}\bar{q}$$

and evaluate the function

$$H(z,\bar{z}) = \langle p, F(x^0 + zq + \bar{z}\bar{q}, r_0) - x^0 \rangle.$$

Computing its Taylor expansion at $(z,\bar{z}) = (0,0)$,

$$H(z,\bar{z}) = e^{i\theta_0} z + \sum_{2 \le j+k \le 3} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(|z|^4),$$

gives

$$g_{20} = -2 + i\frac{2\sqrt{3}}{3}, \quad g_{11} = i\frac{2\sqrt{3}}{3}, \quad g_{02} = 2 + i\frac{2\sqrt{3}}{3}, \quad g_{21} = 0,$$

that allows us to find the critical real part

$$\begin{aligned} a(0) &= \mathrm{Re}\left(\frac{e^{-i\theta_0} g_{21}}{2}\right) - \mathrm{Re}\left(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})} g_{20} g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 \\ &= -2 < 0. \end{aligned}$$

Therefore, a unique and stable closed invariant curve bifurcates from the nontrivial fixed point for $r > 2$ (see Figure 4.11). ◇

4.8 Exercises

(1) Prove that in a small neighborhood of $x = 0$ the number and stability of fixed points and periodic orbits of the maps (4.1) and (4.8) are independent of higher-order terms, provided $|\alpha|$ is sufficiently small. (*Hint*: To prove the absence of long-period cycles, use asymptotic stability arguments.)

(2) Show that the normal form coefficient $c(0)$ for the flip bifurcation (4.11) can be computed in terms of the second iterate of the map:

$$c(0) = -\frac{1}{12} \frac{\partial^3}{\partial x^3} f_\alpha^2(x)\bigg|_{(x,\alpha)=(0,0)},$$

where $f_\alpha(x) = f(x, \alpha)$. (*Hint*: Take into account that $f_x(0,0) = -1$.)

(3) **(Logistic map)** Consider the following map (May [1976]):

$$f_\alpha(x) = \alpha x(1 - x),$$

depending on a single parameter α.

(a) Show that at $\alpha_1 = 3$ the map exhibits the flip bifurcation, namely, a stable fixed point of f_α becomes unstable, while a stable period-two cycle bifurcates from this point for $\alpha > 3$. (*Hint*: Use the formula from Exercise

(a) Show that at $\alpha_1 = 3$ the map exhibits the flip bifurcation, namely, a stable fixed point of f_α becomes unstable, while a stable period-two cycle bifurcates from this point for $\alpha > 3$. (*Hint*: Use the formula from Exercise 2 above.)

(b) Prove that at $\alpha_0 = 1 + \sqrt{8}$ the logistic map has a fold bifurcation generating a stable and an unstable cycle of *period three* as α increases.

(4) (Second period doubling in Ricker's model) Verify that the second period doubling takes place in Ricker's map (4.12) at $\alpha_2 = 12.50925\ldots$. (*Hint*: Introduce $y = \alpha x e^{-x}$ and write a system of three equations for the three unknowns (x, y, α) defining a period-two cycle $\{x, y\}$ with multiplier $\mu = -1$. Use one of the standard routines implementing Newton's method (see Chapter 10) to solve the system numerically starting from some suitable initial data.)

(5) (Henon map) Consider the following invertible planar map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -\epsilon x + \mu - y^2 \end{pmatrix}$$

depending on two parameters. Find a curve in the (ϵ, μ)-plane corresponding to the flip bifurcation of a fixed point.

(6) Derive formula (4.21) for $c_1(0)$ for the Neimark-Sacker bifurcation.

(7) (Discrete-time predator-prey model)
Consider the following discrete-time system (Maynard Smith [1968]):

$$\begin{aligned} x_{k+1} &= \alpha x_k(1 - x_k) - x_k y_k, \\ y_{k+1} &= \frac{1}{\beta} x_k y_k, \end{aligned}$$

which is a discrete-time version of a standard predator-prey model. Here x_k and y_k are the prey and predator numbers, respectively, in year (generation) k, and it is assumed that in the absence of prey the predators become extinct in one generation.

(a) Prove that a nontrivial fixed point of the map undergoes a Neimark-Sacker bifurcation on a curve in the (α, β)-plane, and compute the direction of the closed invariant-curve bifurcation.

(b) Guess what happens to the emergent closed invariant curve for parameter values far from the bifurcation curve.

4.9 Appendix 1: Feigenbaum's universality

As mentioned previously, many one-dimensional, parameter-dependent dynamical systems

$$x \mapsto f_\alpha(x), \quad x \in \mathbb{R}^1, \tag{A1.1}$$

As mentioned previously, many one-dimensional, parameter-dependent dynamical systems

$$x \mapsto f_\alpha(x), \quad x \in \mathbb{R}^1, \tag{A1.1}$$

exhibit infinite cascades of period doublings. Moreover, the corresponding flip bifurcation parameter values, $\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots$, form (asymptotically) a geometric progression:

$$\frac{\alpha_i - \alpha_{i-1}}{\alpha_{i+1} - \alpha_i} \to \mu_F,$$

as $i \to \infty$, where $\mu_F = 4.6692\ldots$ is a system-independent (universal) constant. The sequence $\{\alpha_i\}$ has a limit α_∞. At α_∞ the dynamics of the system become "chaotic," since its orbits become irregular, nonperiodic sequences.

The phenomenon was first explained for special noninvertible dynamical systems $(A1.1)$, that belong for all parameter values to some class $\mathcal{Y}$. Namely, a system

$$x \mapsto f(x) \tag{A1.2}$$

from this class satisfies the following conditions:

(1) $f(x)$ is an even smooth function, $f : [-1, 1] \to [-1, 1]$;
(2) $f'(0) = 0$, $x = 0$ is the only maximum, $f(0) = 1$;
(3) $f(1) = -a < 0$;
(4) $b = f(a) > a$;
(5) $f(b) = f^2(a) < a$;

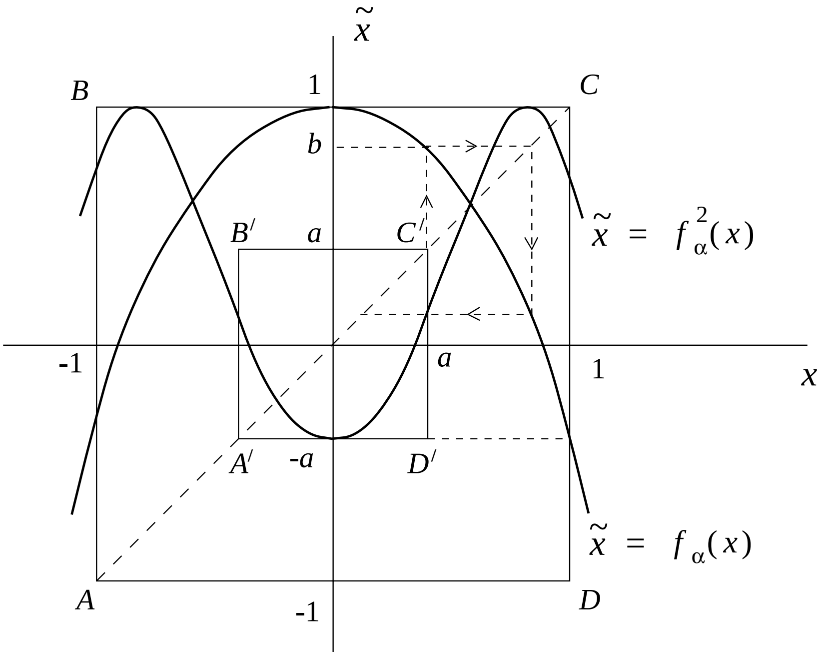


FIGURE 4.12. A map satisfying conditions (1) through (5) and its second iterate.

where a and b are positive (see Figure 4.12). The function $f_\alpha(x) = 1 - \alpha x^2$ is in this class for $\alpha > 1$.

Consider the second iterate f_α^2 of a map satisfying conditions (1) through (5). In the square $A'B'C'D'$ (see Figure 4.12), the graph of f_α^2, after a coordinate dilatation and a sign change, looks similar to the graph of f_α in the unit square $ABCD$. For example, if $f_\alpha(x) = 1 - \alpha x^2$, then $f_\alpha^2(x) =$ $(1 - \alpha) + 2\alpha^2 x^2 + \cdots$. This observation leads to the introduction of a map

(5). In the square $A'B'C'D'$ (see Figure 4.12), the graph of f_α, after a coordinate dilatation and a sign change, looks similar to the graph of f_α in the unit square $ABCD$. For example, if $f_\alpha(x) = 1 - \alpha x^2$, then $f_\alpha^2(x) = (1-\alpha) + 2\alpha^2 x^2 + \cdots$. This observation leads to the introduction of a map defined on functions in $\mathcal{Y}$,

$$(Tf)(x) = -\frac{1}{a} f(f(-ax)), \tag{A1.3}$$

where $a = -f(1)$. Notice that a depends on f.

Definition 4.4 *The map T is called the* doubling operator.

It can be checked that map (A1.3) transforms a function $f \in \mathcal{Y}$ into some function $Tf \in \mathcal{Y}$. Therefore, we can consider a *discrete-time dynamical system* $\{\mathbb{Z}_+, \mathcal{Y}, T^k\}$. This is a dynamical system with the infinite-dimensional state space $\mathcal{Y}$, which is a *function space.* Moreover, the doubling operator is not invertible in general. Thus, we have to consider only *positive* iterations of T.

We shall state the following theorems without proof. They have been proved with the help of a computer and delicate error estimates.

Theorem 4.7 (Fixed-point existence) *The map $T : \mathcal{Y} \to \mathcal{Y}$ defined by* (A1.3) *has a fixed point $\varphi \in \mathcal{Y}$: $T\varphi = \varphi$.* □

It has been found that

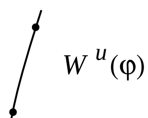
$$\varphi(x) = 1 - 1.52763\ldots x^2 + 0.104815\ldots x^4 + 0.0267057\ldots x^6 + \ldots.$$

In Exercise 1 of Chapter 10 we discuss how to obtain some approximations to $\varphi(x)$.

Theorem 4.8 (Saddle properties of the fixed point) *The linear part L of the doubling operator T at its fixed point φ has only one eigenvalue $\mu_F = 4.6692\ldots$ with $|\mu_F| > 1$. The rest of the spectrum of L is located strictly inside the unit circle.* □

The terms "linear part" and "spectrum" of L are generalizations to the infinite-dimensional case of the notions of the Jacobian matrix and its eigenvalues. An interested reader can find exact definitions in standard textbooks on functional analysis.

Theorems 4.7 and 4.8 mean that the system $\{\mathbb{Z}_+, \mathcal{Y}, T^k\}$ has a saddle fixed point. This fixed point φ (a function that is transformed by the doubling operator into itself) has a codim 1 stable invariant manifold $W^s(\varphi)$ and a one-dimensional unstable invariant manifold $W^u(\varphi)$. The stable manifold is composed by functions $f \in \mathcal{Y}$, which become increasingly similar to φ under iteration of T. The unstable manifold is composed of functions for which *all* their preimages under the action of T remain close to φ. This is a curve in the function space $\mathcal{Y}$ (Figure 4.13 sketches the manifold structure).



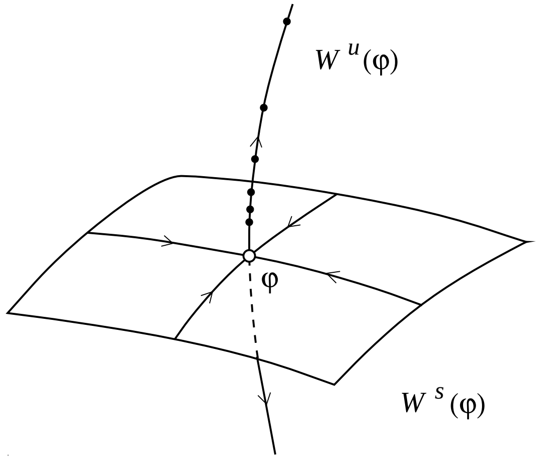


FIGURE 4.13. Stable and unstable manifolds of the fixed point φ.

Notice that maps Tf and f^2 are topologically equivalent (the relevant homeomorphism is a simple scaling; see (A1.3)). Hence, if Tf has a periodic orbit of period N, f^2 has a periodic orbit of the same period and f therefore has a periodic orbit of period $2N$. This simple observation plays the central role in the following. Consider all maps from $\mathcal{Y}$ having a fixed point with multiplier $\mu = -1$. Such maps form a codim 1 manifold $\Sigma \subset \mathcal{Y}$. The following result has also been established with the help of a computer.

Theorem 4.9 (Manifold intersection) *The manifold Σ intersects the unstable manifold $W^u(\varphi)$ transversally.* $\square$

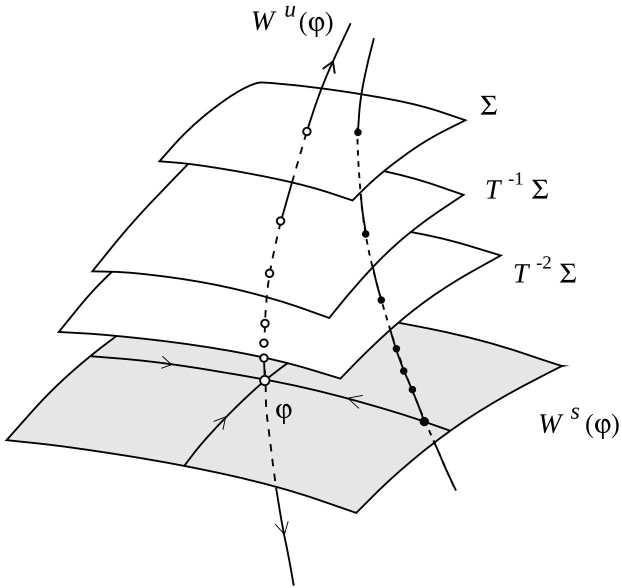


FIGURE 4.14. Preimages of a surface Σ intersecting the unstable manifold $W^u(\varphi)$.

By analogy with a finite-dimensional saddle, it is clear that the preimages $T^{-k}\Sigma$ will accumulate on $W^s(\varphi)$ as $k \to \infty$ (see Figure 4.14). Taking into account the previous observation, we can conclude that $T^{-1}\Sigma$ is composed of maps having a cycle of period two with a multiplier -1, that $T^{-2}\Sigma$ is formed by maps having a cycle of period four with a multiplier -1, and so forth. Any generic one-parameter dynamical system f_α from the considered class corresponds to a curve Λ in $\mathcal{Y}$. If this curve is sufficiently of maps having a cycle of period two with a multiplier -1, that $T^{-2}\Sigma$ is formed by maps having a cycle of period four with a multiplier -1, and so forth. Any generic one-parameter dynamical system f_α from the considered class corresponds to a curve Λ in $\mathcal{Y}$. If this curve is sufficiently close to $W^u(\varphi)$, it will intersect *all* the preimages $T^{-k}\Sigma$. The points of intersection define a sequence of bifurcation parameter values $\alpha_1, \alpha_2, \ldots$ corresponding to a cascade of period doublings. Asymptotic properties of this sequence are clearly determined by the unstable eigenvalue μ_F. Indeed, let ξ be a coordinate along $W^u(\varphi)$, and let ξ_k denote the coordinate of the intersection of $W^u(\varphi)$ with $T^{-k}\Sigma$. The doubling operator *restricted* to the unstable manifold has the form

$$\xi \mapsto \mu_F \xi + O(\xi^2)$$

and is invertible, with the inverse given by

$$\xi \mapsto \frac{1}{\mu_F}\xi + O(\xi^2).$$

Since

$$\xi_{k+1} = \frac{1}{\mu_F}\xi_k + O(\xi_k^2),$$

we have

$$\frac{\xi_k - \xi_{k-1}}{\xi_{k+1} - \xi_k} \to \mu_F$$

as $k \to \infty$, as does the sequence of the bifurcation parameter values on the curve Λ.

4.10 Appendix 2: Proof of Lemma 4.3

In this appendix we prove the following lemma, which is the complex analog of Lemma 4.3.

Lemma 4.8 *The map*

$$\tilde{z} = e^{i\theta(\alpha)} z(1 + \alpha + d(\alpha)|z|^2) + g(z, \bar{z}, \alpha), \tag{A2.1}$$

where $d(\alpha) = a(\alpha) + ib(\alpha)$; $a(\alpha)$, $b(\alpha)$, *and* $\theta(\alpha)$ *are smooth real-valued functions;* $a(0) < 0, 0 < \theta(0) < \pi, g = O(|z|^4)$ *is a smooth complex-valued function of* $z, \bar{z}, \alpha$, *has a stable closed invariant curve for sufficiently small* $\alpha > 0$.

Proof:
Step 1 (*Rescaling and shifting*). First, introduce new variables (s, φ) by the formula

$$z = \sqrt{-\frac{\alpha}{a(\alpha)}} e^{i\varphi}(1 + s). \tag{A2.2}$$

Substitution of (A2.2) into (A2.1) gives

$$\begin{aligned} e^{i\tilde{\varphi}}(1 + \tilde{s}) \quad &= \quad e^{i(\varphi + \theta(\alpha))}(1 + s)\left[1 - \alpha(2s + s^2) + i\alpha\nu(\alpha)(1 + s)^2\right] \\ &\quad + \alpha^{3/2} h(s, \varphi, \alpha), \end{aligned}$$

$$
\begin{aligned}
e^{i\tilde{\varphi}}(1+\tilde{s}) &= e^{i(\varphi+\theta(\alpha))}(1+s)\left[1-\alpha(2s+s^2)+i\alpha\nu(\alpha)(1+s)^2\right] \\
&\quad + \alpha^{3/2}h(s,\varphi,\alpha),
\end{aligned}
$$

where

$$
\nu(\alpha) = -\frac{b(\alpha)}{a(\alpha)},
$$

and h is a smooth complex-valued function of $(s,\varphi,\alpha^{1/2})$. Thus, the map (A2.1) in (s,φ)-coordinates reads

$$
\begin{cases}
\tilde{s} = (1-2\alpha)s - \alpha(3s^2+s^3) + \alpha^{3/2}p(s,\varphi,\alpha), \\
\tilde{\varphi} = \varphi + \theta(\alpha) + \alpha\nu(\alpha)(1+s)^2 + \alpha^{3/2}q(s,\varphi,\alpha),
\end{cases}
\tag{A2.3}
$$

where p, q are smooth real-valued functions of $(s,\varphi,\alpha^{1/2})$. Now apply the scaling

$$
s = \sqrt{\alpha}\xi. \tag{A2.4}
$$

After rescaling accounting to (A2.4), the map (A2.3) takes the form

$$
\begin{cases}
\tilde{\xi} = (1-2\alpha)\xi - \alpha^{3/2}(3\xi^2+\alpha^{1/2}\xi^3) + \alpha p^{(1)}(\xi,\varphi,\alpha), \\
\tilde{\varphi} = \varphi + [\theta(\alpha)+\alpha\nu(\alpha)] + \alpha^{3/2}\nu(\alpha)(2\xi+\alpha^{1/2}\xi^2) + \alpha^{3/2}q^{(1)}(\xi,\varphi,\alpha),
\end{cases}
\tag{A2.5}
$$

where

$$
p^{(1)}(\xi,\varphi,\alpha) = p(\alpha^{1/2}\xi,\varphi,\alpha),\ \ q^{(1)}(\xi,\varphi,\alpha) = q(\alpha^{1/2}\xi,\varphi,\alpha),
$$

are smooth with respect to $(\xi, \varphi, \alpha^{1/2})$. Denote $\omega(\alpha) = \theta(\alpha) + \alpha\nu(\alpha)$, and notice that $p^{(1)}$ can be written as

$$p^{(1)}(\xi, \varphi, \alpha) = r^{(0)}(\varphi, \alpha) + \alpha^{1/2} r^{(1)}(\xi, \varphi, \alpha).$$

Now (A2.5) can be represented by

$$\left\{ \begin{array}{rcl} \tilde{\xi} & = & (1 - 2\alpha)\xi + \alpha r^{(0)}(\varphi, \alpha) + \alpha^{3/2} r^{(2)}(\xi, \varphi, \alpha), \\ \tilde{\varphi} & = & \varphi + \omega(\alpha) + \alpha^{3/2} q^{(2)}(\xi, \varphi, \alpha), \end{array} \right. \tag{A2.6}$$

with

$$\begin{array}{rcl} r^{(2)}(\xi, \varphi, \alpha) & = & -(3\xi^2 + \alpha^{1/2}\xi^3) + r^{(1)}(\xi, \varphi, \alpha), \\ q^{(2)}(\xi, \varphi, \alpha) & = & \nu(\alpha)(2\xi + \alpha^{1/2}\xi^2) + q^{(1)}(\xi, \varphi, \alpha). \end{array}$$

The functions $r^{(2)}$ and $q^{(2)}$ have the same smoothness as $p^{(1)}$ and $q^{(1)}$. Finally, perform a coordinate shift, eliminating the term $\alpha r^{(0)}(\varphi, \alpha)$ from the first equation in (A2.6):

$$\xi = u + \tfrac{1}{2} r^{(0)}(\varphi, \alpha). \tag{A2.7}$$

This gives a map F, which we will work with from now on,

$$F : \left\{ \begin{array}{rcl} \tilde{u} & = & (1 - 2\alpha)u + \alpha^{3/2} H_\alpha(u, \varphi), \\ \tilde{\varphi} & = & \varphi + \omega(\alpha) + \alpha^{3/2} K_\alpha(u, \varphi), \end{array} \right. \tag{A2.8}$$

$$F:\begin{cases}\tilde{u} &= (1-2\alpha)u+\alpha^{3/2}H_\alpha(u,\varphi),\\ \tilde{\varphi} &= \varphi+\omega(\alpha)+\alpha^{3/2}K_\alpha(u,\varphi),\end{cases}\tag{A2.8}$$

where $\omega(\alpha)$ is smooth and

$$\begin{aligned}H_\alpha(u,\varphi) &= r^{(2)}\left(u+\tfrac{1}{2}r^{(0)}(\varphi,\alpha),\varphi,\alpha\right),\\ K_\alpha(u,\varphi) &= q^{(2)}\left(u+\tfrac{1}{2}r^{(0)}(\varphi,\alpha),\varphi,\alpha\right),\end{aligned}$$

are smooth functions of $(u,\varphi,\alpha^{1/2})$ that are 2π-periodic in φ.

Notice that the band $\{(u,\varphi): |u|\le 1, \varphi\in[0,2\pi]\}$ corresponds to a band of $O(\alpha)$ width around the circle

$$S_0(\alpha)=\left\{z: |z|^2=-\frac{\alpha}{a(\alpha)}\right\}$$

in (A2.1), which has an $O(\alpha^{1/2})$ radius in the original coordinate z. In what follows, it is conveinient to introduce a number

$$\lambda=\sup_{|u|\le 1,\varphi\in[0,2\pi]}\left\{|H_\alpha|,|K_\alpha|,\left|\frac{\partial H_\alpha}{\partial u}\right|,\left|\frac{\partial K_\alpha}{\partial u}\right|,\left|\frac{\partial H_\alpha}{\partial \varphi}\right|,\left|\frac{\partial K_\alpha}{\partial \varphi}\right|\right\}.\tag{A2.9}$$

So defined, λ depends on α but remains bounded as $\alpha\to 0$.

Step 2 (*Definition of the function space*). We will characterize the closed curves by elements of a function space U. By definition, $u\in U$ is a 2π-periodic function $u=u(\varphi)$ satisfying the following two conditions:

(U.1) $|u(\varphi)| \le 1$ for all φ;
(U.2) $|u(\varphi_1) - u(\varphi_2)| \le |\varphi_1 - \varphi_2|$ for all φ_1, φ_2.

The first property means that $u(\varphi)$ is *absolutely bounded* by unity, while the second means that $u(\varphi)$ is *Lipschitz continuous* with Lipschitz constant equal to one. Space U is a complete metric space with respect to the norm

$$\|u\| = \sup_{\varphi \in [0,2\pi]} |u(\varphi)|.$$

Recall from Chapter 1 that a map $\mathcal{F} : U \to U$ (transforming a function $u(\varphi) \in U$ into some other function $\tilde{u}(\varphi) = (\mathcal{F}u)(\varphi) \in U$) is a *contraction* if there is a number $\varepsilon,\ 0 < \varepsilon < 1$, such that

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\| \le \varepsilon \|u_1 - u_2\|$$

for all $u_{1,2} \in U$. A contraction map in a complete normed space has a unique fixed point $u^{(\infty)} \in U$:

$$\mathcal{F}(u^{(\infty)}) = u^{(\infty)}.$$

Moreover, the fixed point $u^{(\infty)}$ is a globally stable equilibrium of the infinite-dimensional dynamical system $\{U, \mathcal{F}\}$, that is,

$$\lim_{k \to +\infty} \|\mathcal{F}^k(u) - u^{(\infty)}\| = 0,$$

$$\lim_{k \to +\infty} \|\mathcal{F}^k(u) - u^{(\infty)}\| = 0,$$

for all $u \in U$ (see Figure 4.15). The above two facts are often referred to as the Contraction Mapping Principle.

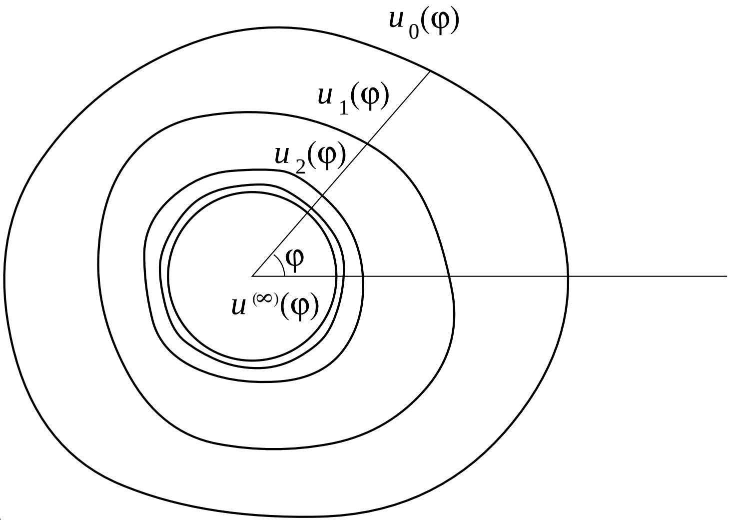


FIGURE 4.15. Accumulating closed curves.

Step 3 (*Construction of the map* $\mathcal{F}$). We will consider a map $\mathcal{F}$ *induced* by F on U. This means that if u represents a closed curve, then $\tilde{u} = \mathcal{F}(u)$ represents its image under the map F defined by (A2.8).

Suppose that a function $u = u(\varphi)$ from U is given. To construct the map $\mathcal{F}$, we have to specify a procedure for each given φ that allows us to find the corresponding $\tilde{u}(\varphi) = (\mathcal{F}u)(\varphi)$. Notice, however, that F is nearly a *rotation* by the angle $\omega(\alpha)$ in φ. Thus, a point $(\tilde{u}(\varphi), \varphi)$ in the resulting curve is the image of a point $(u(\hat{\varphi}), \hat{\varphi})$ in the original curve with a *different* angle coordinate $\hat{\varphi}$ (see Figure 4.16).

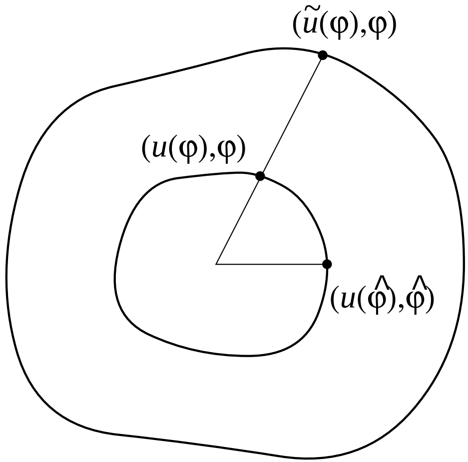


FIGURE 4.16. Definition of the map.

To show that $\hat{\varphi}$ is uniquely defined, we have to prove that the equation

$$\varphi = \hat{\varphi} + \omega(\alpha) + \alpha^{3/2} K_\alpha(u(\hat{\varphi}), \hat{\varphi}) \qquad \text{(A2.10)}$$

has a unique solution $\hat{\varphi} = \hat{\varphi}(\varphi)$ for any given $u \in U$. This is the case, since

$$\varphi = \hat{\varphi} + \omega(\alpha) + \alpha^{3/2} K_\alpha(u(\hat{\varphi}), \hat{\varphi}) \tag{A2.10}$$

has a unique solution $\hat{\varphi} = \hat{\varphi}(\varphi)$ for any given $u \in U$. This is the case, since the right-hand side of (A2.10) is a strictly increasing function of $\hat{\varphi}$. Indeed, let $\varphi_2 > \varphi_1$; then, according to (A2.8),

$$\begin{aligned}\tilde{\varphi_2} - \tilde{\varphi_1} &= \varphi_2 - \varphi_1 + \alpha^{3/2}\left[K_\alpha(u(\varphi_2), \varphi_2) - K_\alpha((\varphi_1), \varphi_1)\right] \\ &\geq \varphi_2 - \varphi_1 - \alpha^{3/2}\left|K_\alpha(u(\varphi_2), \varphi_2) - K_\alpha((\varphi_1), \varphi_1)\right|.\end{aligned}$$

Taking into account that K_α is a smooth function with (A2.9) and (U.2), we get

$$\begin{aligned}|K_\alpha(u(\varphi_2), \varphi_2) - K_\alpha(u(\varphi_1), \varphi_1)| &\leq \lambda[|u(\varphi_2) - u(\varphi_1)| + |\varphi_2 - \varphi_1|] \\ &\leq 2\lambda|\varphi_2 - \varphi_1| = 2\lambda(\varphi_2 - \varphi_1).\end{aligned}$$

This last estimate can also be written as

$$-\left|K_\alpha(u(\varphi_2), \varphi_2) - K_\alpha((\varphi_1), \varphi_1)\right| \geq -2\lambda(\varphi_2 - \varphi_1),$$

which implies

$$\tilde{\varphi_2} - \tilde{\varphi_1} \geq (1 - 2\lambda\alpha^{3/2})(\varphi_2 - \varphi_1).$$

Thus, the right-hand side of (A2.10) is a strictly increasing function, provided α is small enough, and its solution $\hat{\varphi}$ is uniquely defined.[2] From the

[2]Meanwhile, $\hat{\varphi} \approx \varphi - \omega(\alpha)$.

above estimates, it also follows that $\hat{\varphi}(\varphi)$ – that is, the inverse function to the function given by (A2.10) – is Lipschitz continuous:

$$|\hat{\varphi}(\varphi_1) - \hat{\varphi}(\varphi_2)| \leq (1 - 2\lambda\alpha^{3/2})^{-1}|\varphi_1 - \varphi_2|. \tag{A2.11}$$

Now we can define the map $\tilde{u} = \mathcal{F}(u)$ by the formula

$$\tilde{u}(\varphi) = (1 - 2\alpha)u(\hat{\varphi}) + \alpha^{3/2}K_\alpha(u(\hat{\varphi}), \hat{\varphi}), \tag{A2.12}$$

where $\hat{\varphi}$ is the solution of (A2.10). The mere definition, of course, is not enough and we have to verify that $\mathcal{F}(u) \in U$, if $u \in U$, namely, to check (U.1) and (U.2) for $\tilde{u} = \mathcal{F}(u)$.

Condition (U.1) for $\tilde{u}$ follows from the estimate

$$|\tilde{u}(\varphi)| \leq (1 - 2\alpha)|u(\hat{\varphi})| + \alpha^{3/2}|H_\alpha(u(\hat{\varphi}), \hat{\varphi})| \leq 1 - 2\alpha + \lambda\alpha^{3/2},$$

where we have used (U.1) for u and the definition (A2.9) of λ. Thus, $|\tilde{u}| \leq 1$ if α is small enough and positive. Condition (U.2) for $\tilde{u}$ is obtained by the sequence of estimates:

$$\begin{aligned} |\tilde{u}(\varphi_1) - \tilde{u}(\varphi_2)| &\leq (1 - 2\alpha)|u(\hat{\varphi}_1) - u(\hat{\varphi}_2)| \\ &\quad + \alpha^{3/2}|H_\alpha(u(\hat{\varphi}_1), \hat{\varphi}_1) - H_\alpha(u(\hat{\varphi}_2), \hat{\varphi}_2)| \\ &\leq (1 - 2\alpha)|u(\hat{\varphi}_1) - u(\hat{\varphi}_2)| \\ &\quad + \alpha^{3/2}\lambda\Big[|\tilde{u}(\varphi_1) - \tilde{u}(\varphi_2)| + |\hat{\varphi}_1 - \hat{\varphi}_2|\Big] \\ &\leq (1 - 2\alpha + 2\lambda\alpha^{3/2})|\hat{\varphi}_1 - \hat{\varphi}_2| \end{aligned}$$

$$
\begin{aligned}
&+ \alpha^{3/2}|H_\alpha(u(\hat\varphi_1),\hat\varphi_1) - H_\alpha(u(\hat\varphi_2),\hat\varphi_2)| \\
\leq\ & (1-2\alpha)|u(\hat\varphi_1) - u(\hat\varphi_2)| \\
&+ \alpha^{3/2}\lambda\Big[|\tilde u(\varphi_1) - \tilde u(\varphi_2)| + |\hat\varphi_1 - \hat\varphi_2|\Big] \\
\leq\ & (1-2\alpha+2\lambda\alpha^{3/2})|\hat\varphi_1 - \hat\varphi_2|,
\end{aligned}
$$

where the final inequality holds due to the Lipschitz continuity of u. Inserting the estimate (A2.11), we get

$$
|\tilde u(\varphi_1) - \tilde u(\varphi_2)| \leq (1-2\alpha+2\lambda\alpha^{3/2})(1-2\lambda\alpha^{3/2})^{-1}|\varphi_1 - \varphi_2|.
$$

Thus, (U.2) also holds for $\tilde u$ for all sufficiently small positive α. Therefore, the map $\tilde u = \mathcal{F}(u)$ is well defined.

Step 4 (*Verification of the contraction property*). Now suppose two functions $u_1, u_2 \in U$ are given. What we need to obtain is the estimation of $\|\tilde u_1 - \tilde u_2\|$ in terms of $\|u_1 - u_2\|$. By the definition (A2.12) of $\tilde u = \mathcal{F}(u)$,

$$
\begin{aligned}
\|\tilde u_1(\varphi) - \tilde u_2(\varphi))\| \leq\ & (1-2\alpha)|u_1(\hat\varphi_1) - u_2(\hat\varphi_2)| \\
&+ \alpha^{3/2}|H_\alpha(u_1(\hat\varphi_1),\hat\varphi_1) - H_\alpha(u_2(\hat\varphi_2),\hat\varphi_2)| \\
\leq\ & (1-2\alpha)|u_1(\hat\varphi_1) - u_2(\hat\varphi_2)| \\
&+ \alpha^{3/2}\lambda\Big[|u_1(\hat\varphi_1) - u_2(\hat\varphi_2)| + |\hat\varphi_1 - \hat\varphi_2|\Big],
\end{aligned}
\tag{A2.13}
$$

where $\hat\varphi_1$ and $\hat\varphi_2$ are the unique solutions of

$$
\varphi = \hat\varphi_1 + \omega(\alpha) + \alpha^{3/2}K_\alpha(u_1(\hat\varphi_1),\hat\varphi_1) \tag{A2.14}
$$

and

$$\varphi = \hat{\varphi}_2 + \omega(\alpha) + \alpha^{3/2} K_\alpha(u_2(\hat{\varphi}_2), \hat{\varphi}_2), \tag{A2.15}$$

respectively. The estimates (A2.13) have not solved the problem yet, since we have to use only $\|u_1 - u_2\|$ in the right-hand side. First, express $|u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)|$ in terms of $\|u_1 - u_2\|$ *and* $|\hat{\varphi}_1 - \hat{\varphi}_2|$:

$$\begin{aligned} |u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| &= |u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_1) + u_2(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| \\ &\le |u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_1)| + |u_2(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| \\ &\le \|u_1 - u_2\| + |\hat{\varphi}_1 - \hat{\varphi}_2|. \end{aligned} \tag{A2.16}$$

The last inequality has been obtained using the definition of the norm and the Lipschitz continuity of u_2. To complete the estimates, we need to express $|\hat{\varphi}_1 - \hat{\varphi}_2|$ in terms of $\|u_1 - u_2\|$. Subtracting (A2.15) from (A2.14), transposing, and taking absolute values yield

$$\begin{aligned} |\hat{\varphi}_1 - \hat{\varphi}_2| &\le \alpha^{3/2} |K_\alpha(u_1(\hat{\varphi}_1), \hat{\varphi}_1) - K_\alpha(u_2(\hat{\varphi}_2), \hat{\varphi}_2)| \\ &\le \alpha^{3/2} \lambda [|u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| + |\hat{\varphi}_1 - \hat{\varphi}_2|]. \end{aligned}$$

Inserting (A2.16) into this inequality and collecting all the terms involving $|\hat{\varphi}_1 - \hat{\varphi}_2|$ on the left, result in

$$|\hat{\varphi}_1 - \hat{\varphi}_2| \le (1 - 2\alpha^{3/2}\lambda)^{-1} \alpha^{3/2} \lambda \|u_1 - u_2\|. \tag{A2.17}$$

Using the estimates (A2.16) and (A2.17), we can complete (A2.13) as fol-

$$|\hat{\varphi}_1 - \hat{\varphi}_2| \leq (1 - 2\alpha^{3/2}\lambda)^{-1}\alpha^{3/2}\lambda\|u_1 - u_2\|. \tag{A2.17}$$

Using the estimates (A2.16) and (A2.17), we can complete (A2.13) as follows:

$$\|\tilde{u}_1(\varphi) - \tilde{u}_2(\varphi))\| \leq \epsilon\|u_1 - u_2\|,$$

where

$$\epsilon = (1-2\alpha)\left[1 + \alpha^{3/2}\lambda(1 - 2\alpha^{3/2}\lambda)^{-1}\right] + \alpha^{3/2}\lambda\left[1 + 2\alpha^{3/2}\lambda(1 - 2\alpha^{3/2}\lambda)^{-1}\right].$$

Since

$$\epsilon = 1 - 2\alpha + O(\alpha^{3/2}),$$

the map $\mathcal{F}$ is a contraction in U for small positive α. Therfore, it has a unique stable fixed point $u^{(\infty)} \in U$.

Step 5 (*Stability of the invariant curve*). Now take a point (u_0, φ_0) within the band $\{(u, \varphi) : |u| \leq 1, \varphi \in [0, 2\pi]\}$. If the point belongs to the curve given by $u^{(\infty)}$, it remains on this curve under iterations of F, since the map $\mathcal{F}$ maps this curve into itself. If the point does not lie on the invariant curve, take some (noninvariant) closed curve passing through it represented by $u^{(0)} \in U$, say. Such a curve always exists. Let us apply the iterations of the map F defined by (A2.8) to this point. We get a sequence of points

$$\{(u_k, \varphi_k)\}_{k=0}^{\infty}.$$

It is clear that each point from this sequence belongs to the corresponding iterate of the curve $u^{(0)}$ under the map $\mathcal{F}$. We have just shown that the iterations of the curve converge to the invariant curve given by $u^{(\infty)}$. Therefore, the point sequence must also converge to the curve. This proves the stability of the closed invariant curve as the invariant set of the map and completes the proof. $\square$

4.11 Appendix 3: Bibliographical notes

The dynamics generated by one-dimensional maps is a classical mathematical subject, studied in detail (see Whitley [1983] and van Strien [1991] for surveys). Properties of the fixed points and period-two cycles involved in the fold and flip bifurcations were known long ago. Explicit formulation of the topological normal form theorems for these bifurcations is due to Arnold [1983]. A complete proof, that the truncation of the higher-order terms in the normal forms results in locally topologically equivalent systems, happens to be unexpectedly difficult (see Newhouse, Palis & Takens [1983], Arnol'd et al. [1994]) and remains unpublished.

The appearance of a closed invariant curve surrounding a fixed point while a pair of complex multipliers crosses the unit circle was known to Andronov and studied by Neimark [1959] (without explicit statement of all the genericity conditions). A complete proof was given by Sacker [1965], who discovered the bifurcation independently. It became widely known as while a pair of complex multipliers crosses the unit circle was known to Andronov and studied by Neimark [1959] (without explicit statement of all the genericity conditions). A complete proof was given by Sacker [1965], who discovered the bifurcation independently. It became widely known as "Hopf bifurcation for maps" after Ruelle & Takens [1971] and Marsden & McCracken [1976]. A modern treatment of the Neimark-Sacker bifurcation for planar maps can be found in Iooss [1979], where the normal form coefficient $a(0)$ is computed (see also Wan [1978*b*]). In our Appendix 2 we follow, essentially, the proof given in Marsden & McCracken [1976].

The normal form theory for maps is presented by Arnold [1983]. In our analysis of the codimension-one bifurcations of fixed points we need only a small portion of this theory which we develop "on-line."

Cascades of period doubling bifurcations were observed by mathematical ecologists in one-dimensional discrete-time population models (Shapiro [1974] analyzed a model by Ricker [1954], while May [1974] used the logistic map). Feigenbaum [1978] discovered the universality in such cascades and explained its mechanism based on the properties of the doubling operator. The relevant theorems were proved by Lanford [1980] with the help of a computer and delicate error estimates (see also Collet & Eckmann [1980], Babenko & Petrovich [1983]). Feigenbaum-type universality is also proved for some classes of multidimensional discrete-time dynamical systems.

Both the delayed logistic and discrete-time predator-prey models originate in a book by Maynard Smith [1968]. The fate of the closed invariant curve while a parameter "moves" away from the Neimark-Sacker bifurcation was analyzed for the delayed logistic map by Aronson, Chory, Hall &

McGehee [1982].

5

Bifurcations of Equilibria and Periodic Orbits in n-Dimensional Dynamical Systems

In the previous two chapters we studied bifurcations of equilibria and fixed points in generic one-parameter dynamical systems having the *minimum possible* phase dimensions. Indeed, the systems we analyzed were either one- or two-dimensional. This chapter shows that the corresponding bipoints in generic one-parameter dynamical systems having the *minimum possible* phase dimensions. Indeed, the systems we analyzed were either one- or two-dimensional. This chapter shows that the corresponding bifurcations occur in "essentially" the same way for generic n-dimensional systems. As we shall see, there are certain parameter-dependent one- or two-dimensional *invariant manifolds* on which the system exhibits the corresponding bifurcations, while the behavior off the manifolds is somehow "trivial," for example, the manifolds may be exponentially attractive. Moreover, such manifolds (called *center manifolds*) exist for many dissipative infinite-dimensional dynamical systems. Below we derive explicit formulas for the approximation of center manifolds in finite dimensions and for systems restricted to them at bifurcation parameter values. In Appendix 1 we consider a reaction-diffusion system on an interval to illustrate the necessary modifications of the technique to handle infinite-dimensional systems.

5.1 Center manifold theorems

We are going to formulate without proof the main theorems that allow us to reduce the dimension of a given system near a local bifurcation. Let us start with the *critical* case; we assume in this section that the parameters of the system are fixed at their bifurcation values, which are those values for which there is a nonhyperbolic equilibrium (fixed point). We will treat continuous- and discrete-time cases separately.

5.1.1 *Center manifolds in continuous-time systems*

Consider a continuous-time dynamical system defined by

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{5.1}$$

where f is sufficiently smooth, $f(0) = 0$. Let the eigenvalues of the Jacobian matrix A evaluated at the equilibrium point $x_0 = 0$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Suppose the equilibrium is not hyperbolic and that there are thus eigenvalues with zero real part. Assume that there are n_+ eigenvalues (counting multiplicities) with Re $\lambda > 0$, n_0 eigenvalues with Re $\lambda = 0$, and n_- eigenvalues with Re $\lambda < 0$ (see Figure 5.1). Let T^c denote the linear (generalized)

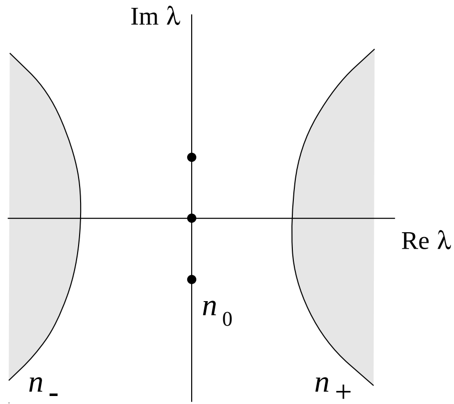


FIGURE 5.1. Critical eigenvalues of an equilibrium.

FIGURE 5.1. Critical eigenvalues of an equilibrium.

eigenspace of A corresponding to the union of the n_0 eigenvalues on the imaginary axis. The eigenvalues with $\mathrm{Re}\,\lambda = 0$ are often called *critical*, as is the eigenspace T^c. Let φ^t denote the flow associated with (5.1). Under the assumptions stated above, the following theorem holds.

Theorem 5.1 (Center Manifold Theorem) *There is a locally defined smooth n_0-dimensional invariant manifold $W^c_{\rm loc}(0)$ of (5.1) that is tangent to T^c at $x = 0$.*

Moreover, there is a neighborhood U of $x_0 = 0$, such that if $\varphi^t x \in U$ for all $t \geq 0 (t \leq 0)$, then $\varphi^t x \to W^c_{\rm loc}(0)$ for $t \to +\infty$ $(t \to -\infty)$. □

Definition 5.1 *The manifold $W^c_{\rm loc}$ is called the* center manifold.

We are not going to present the proof here. If $n_+ = 0$, the manifold $W^c_{\rm loc}$ can be constructed as a local limit of iterations of T^c under φ^1. From now on, we drop the subscript "loc" in order to simplify notation. Figures 5.2 and 5.3 illustrate the theorem for the fold bifurcation on the plane $(n = 2, n_0 = 1, n_- = 1)$ and for the Hopf bifurcation in $\mathbb{R}^3$ $(n = 3, n_0 = 2, n_- = 1)$. In the first case, the center manifold W^c is tangent to the eigenvector corresponding to $\lambda_1 = 0$, while in the second case, it is tangent to a plane spanned by the real and imaginary parts of the complex eigenvector corresponding to $\lambda_1 = i\omega_0,\ \omega_0 > 0$.

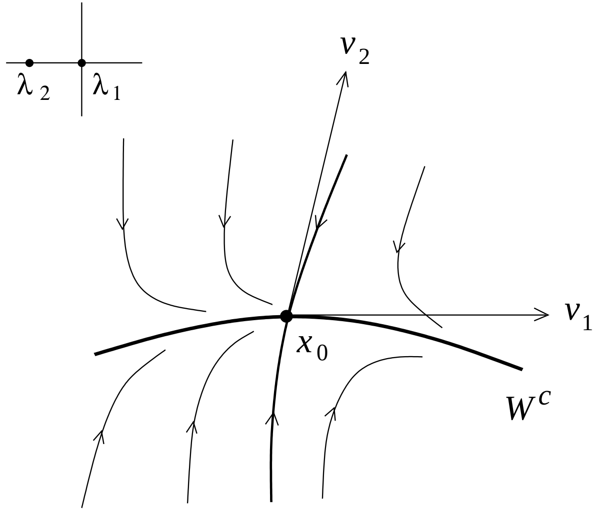
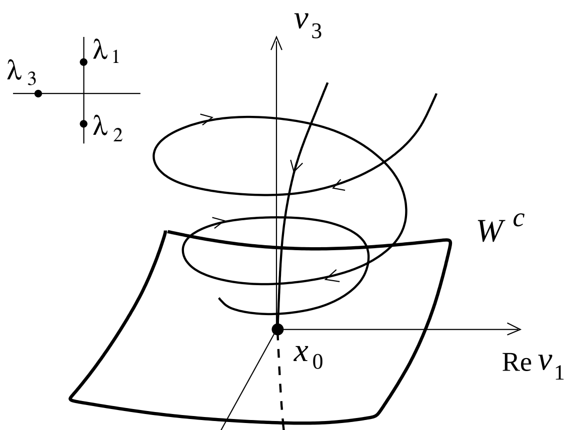


FIGURE 5.2. One-dimensional center manifold at the fold bifurcation.



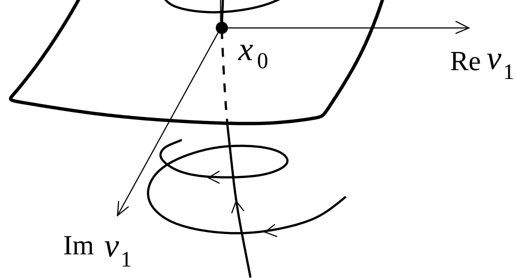


FIGURE 5.3. Two-dimensional center manifold at the Hopf bifurcation.

Remarks:

(1) The second statement of the theorem means that orbits staying near the equilibrium for $t \geq 0$ or $t \leq 0$ tend to W^c in the corresponding time direction. If we know a priori that *all* orbits starting in U remain in this region forever (a necessary condition for this is $n_+ = 0$), then the theorem implies that these orbits approach $W^c(0)$ as $t \to +\infty$. In this case the manifold is "attracting."

(2) W^c *need not* be unique. The system

$$\begin{cases} \dot{x} = x^2, \\ \dot{y} = -y, \end{cases}$$

has an equilibrium $(x, y) = (0, 0)$ with $\lambda_1 = 0, \lambda_2 = -1$ (a fold case). It possesses a family of one-dimensional center manifolds:

$$W^c_\beta(0) = \{(x, y) \ : \ y = \psi_\beta(x)\},$$

where

$$\psi_\beta(x) = \begin{cases} \beta \exp\left(\frac{1}{x}\right) & \text{for } x < 0, \\ 0 & \text{for } x \geq 0, \end{cases}$$

(see Figure 5.4(a)). The system

$$\begin{cases} \dot{x} &= -y - x(x^2 + y^2), \\ \dot{y} &= x - y(x^2 + y^2), \\ \dot{z} &= -z, \end{cases}$$

has an equilibrium $(x, y, z) = (0, 0, 0)$ with $\lambda_{1,2} = \pm i,\ \lambda_3 = -1$ (Hopf case). There is a family of two-dimensional center manifolds in the system given

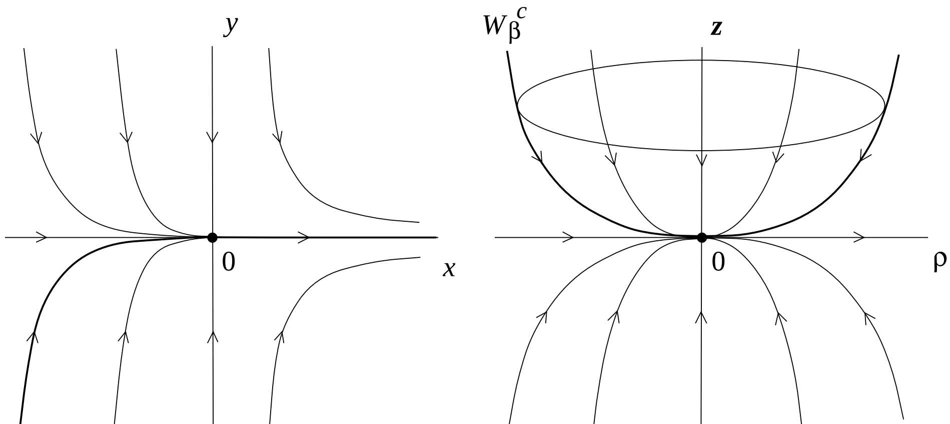


FIGURE 5.4. Nonuniqueness of the center manifold at (a) fold and (b) Hopf bifurcations.

by

$$W^c_\beta(0) = \{(x, y, z) \ : \ z = \phi_\beta(x, y)\},$$

where

$$\phi_\beta(x,y) = \begin{cases} \beta \exp\left(-\frac{1}{2(x^2+y^2)}\right) & \text{for } x^2 + y^2 > 0, \\ 0 & \text{for } x = y = 0, \end{cases}$$

(see Figure 5.4(b)). As we shall see, this nonuniqueness is actually irrelevant for applications.

(3) A center manifold W^c has the same *finite* smoothness as f (if $f \in C^k$ with finite k, W^c is also a C^k manifold) in some neighborhood U of x_0. However, as $k \to \infty$ the neighborhood U may shrink, thus resulting in the nonexistence of a C^∞ manifold W^c for some C^∞ systems (see Exercise 1). $\diamondsuit$

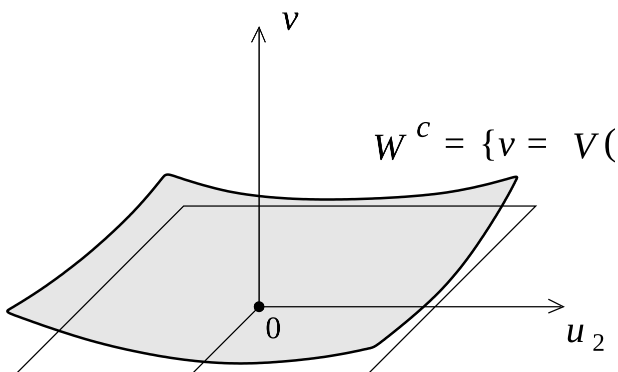
In its eigenbasis,[1] system (5.1) can be written as

$$\begin{cases} \dot{u} & = & Bu + g(u, v), \\ \dot{v} & = & Cv + h(u, v), \end{cases} \tag{5.2}$$

where $u \in \mathbb{R}^{n_0}, v \in \mathbb{R}^{n_+ + n_-}$, B is an $n_0 \times n_0$ matrix with all its n_0 eigenvalues on the imaginary axis, while C is an $(n_+ + n_-) \times (n_+ + n_-)$ matrix with no eigenvalue on the imaginary axis. Functions g and h have Taylor expansions starting with at least quadratic terms. The center manifold W^c of system (5.2) can be locally represented as a graph of a smooth function:

$$W^c = \{(u, v) \ : \ v = V(u)\}$$

(see Figure 5.5). Here $V : \mathbb{R}^{n_0} \to \mathbb{R}^{n_+ + n_-}$, and due to the tangent property of W^c, $V(u) = O(\|u\|^2)$.



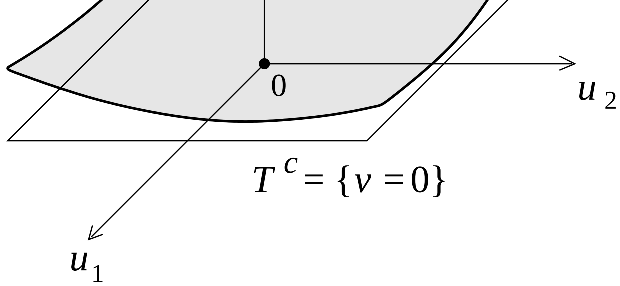


FIGURE 5.5. Center manifold as the graph of a function $v = V(u)$.

Theorem 5.2 (Reduction Principle) *System* (5.2) *is locally topologically equivalent near the origin to the system*

$$\begin{cases} \dot{u} &= Bu + g(u, V(u)), \\ \dot{v} &= Cv. \quad \square \end{cases} \tag{5.3}$$

Notice that the equations for u and v are uncoupled in (5.3). The first equation is the *restriction* of (5.2) to its center manifold. Thus, the dynamics of the structurally unstable system (5.2) are essentially determined

[1]Recall that the *eigenbasis* is a basis formed by all (generalized) eigenvectors of A (or their linear combinations if the corresponding eigenvalues are complex). Actually, the basis used in the following may not be the true eigenbasis: Any basis in the noncritical eigenspace is allowed. In other words, the matrix C may not be in real canonical (Jordan) form.

by this restriction, since the second equation in (5.3) is linear and has exponentially decaying/growing solutions. For example, if $u = 0$ is the asymptotically stable equilibrium of the restriction and all eigenvalues of C have negative real part, then $(u, v) = (0, 0)$ is the asymptotically stable equilibrium of (5.2). Clearly, the dynamics on the center manifold are determined not only by the linear but also by the *nonlinear* terms of (5.2). If there is more than one center manifold, then all the resulting systems (5.3) with different V are locally topologically equivalent.

The second equation in (5.3) can be replaced by the equations of a *standard saddle*:

$$\begin{cases} \dot{v} = -v, \\ \dot{w} = w, \end{cases} \tag{5.4}$$

with $(v, w) \in \mathbb{R}^{n_-} \times \mathbb{R}^{n_+}$. Therefore, the Reduction Principle can be expressed neatly in the following way: *Near a nonhyperbolic equilibrium the system is locally topologically equivalent to the suspension of its restriction to the center manifold by the standard saddle.*

5.1.2 *Center manifolds in discrete-time systems*

Consider now a discrete-time dynamical system defined by

$$x \mapsto f(x), \quad x \in \mathbb{R}^n, \tag{5.5}$$

$$x \mapsto f(x), \quad x \in \mathbb{R}^n, \tag{5.5}$$

where f is sufficiently smooth, $f(0) = 0$. Let the eigenvalues of the Jacobian matrix A evaluated at the fixed point $x_0 = 0$ be $\mu_1, \mu_2, \ldots, \mu_n$. Recall, that we call them *multipliers*. Suppose that the equilibrium is not hyperbolic and there are therefore multipliers on the unit circle (with absolute value one). Assume that there are n_+ multipliers outside the unit circle, n_0 multipliers on the unit circle, and n_- multipliers inside the unit circle (see Figure 5.6). Let T^c denote the linear invariant (generalized) eigenspace of

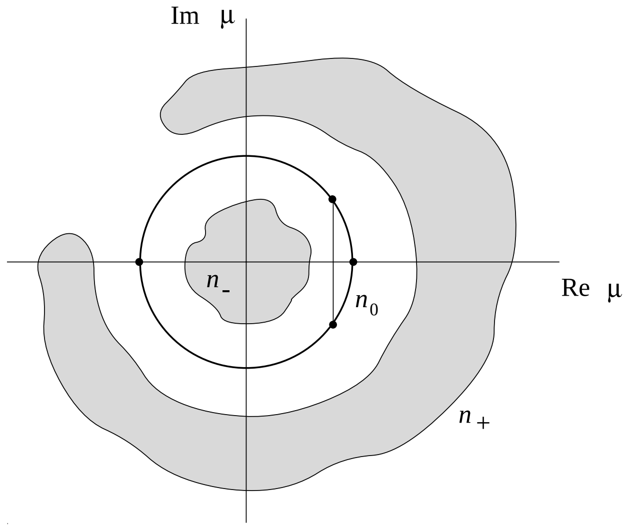


FIGURE 5.6. Critical multipliers of a fixed point.

A corresponding to the union of n_0 multipliers on the unit circle. Then, Theorem 5.1 holds verbatim for system (5.5), if we consider only integer time values and set $\varphi^k = f^k$, the kth iterate of f. Using an eigenbasis, we can rewrite the system as

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} Bu + g(u,v) \\ Cv + h(u,v) \end{pmatrix} \tag{5.6}$$

with the same notation as before, but the matrix B now has eigenvalues on the unit circle, while all the eigenvalues of C are inside and/or outside it. The center manifold possesses the local representation $v = V(u)$, and the Reduction Principle remains valid.

Theorem 5.3 *System* (5.6) *is locally topologically equivalent near the origin to the system*

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} Bu + g(u,V(u)) \\ Cv \end{pmatrix}. \quad \square \tag{5.7}$$

The construction of the standard saddle is more involved for the discrete-time case, since we have to take into account the *orientation* properties of the map in the expanding and contracting directions. First, suppose for simplicity that there are no multipliers outside the unit circle, (i.e., $n_+ = 0$). Then, if $\det C > 0$, the map $v \mapsto Cv$ in (5.7) can be replaced by

$$v \mapsto \frac{1}{2} v$$

simplicity that there are no multipliers outside the unit circle, (i.e., $n_+ = 0$). Then, if $\det C > 0$, the map $v \mapsto Cv$ in (5.7) can be replaced by

$$v \quad \mapsto \quad \tfrac{1}{2}v,$$

which is a *standard orientation-preserving stable node*. However, if $\det C < 0$, the map $v \mapsto Cv$ in (5.7) must be substituted by

$$\begin{cases} v_1 & \mapsto & \tfrac{1}{2}v_1, \\ v_2 & \mapsto & -\tfrac{1}{2}v_2, \end{cases}$$

where $v_1 \in \mathbb{R}^{n_- - 1}$, $v_2 \in \mathbb{R}^1$, which is a *standard orientation-reversing stable node*. If there are now n_+ multipliers outside the unit circle, the *standard unstable node* $w \mapsto \tilde{w},\ w, \tilde{w} \in \mathbb{R}^{n_+}$, should be added to (5.7). The standard unstable node is defined similarly to the standard stable node but with multiplier 2 instead of $\frac{1}{2}$. Standard stable and unstable nodes together define the *standard saddle* map on $\mathbb{R}^{n_- + n_+}$.

5.2 Center manifolds in parameter-dependent systems

Consider now a smooth continuous-time system that depends smoothly on a parameter:

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^1. \tag{5.8}$$

Suppose that at $\alpha = 0$ the system has a nonhyperbolic equilibrium $x = 0$ with n_0 eigenvalues on the imaginary axis and $(n - n_0)$ eigenvalues with nonzero real parts. Let n_- of them have negative real parts, while n_+ have positive real parts. Consider the *extended system*:

$$\begin{cases} \dot{\alpha} &= 0, \\ \dot{x} &= f(x, \alpha). \end{cases} \tag{5.9}$$

Notice that the extended system (5.9) may be nonlinear, even if the original system (5.8) is linear. The Jacobian of (5.9) at the equilibrium $(\alpha, x) = (0, 0)$ is the $(n+1) \times (n+1)$ matrix

$$J = \begin{pmatrix} 0 & 0 \\ f_\alpha(0,0) & f_x(0,0) \end{pmatrix},$$

having $(n_0 + 1)$ eigenvalues on the imaginary axis and $(n - n_0)$ eigenvalues with nonzero real part. Thus, we can apply the Center Manifold Theorem to system (5.9). The theorem guarantees the existence of a center manifold $\mathcal{W}^c \subset \mathbb{R}^1 \times \mathbb{R}^n$, $\dim \mathcal{W}^c = n_0 + 1$. This manifold is tangent at the origin to the (generalized) eigenspace of J corresponding to $(n_0 + 1)$ eigenvalues with zero real part. Since $\dot{\alpha} = 0$, the hyperplanes $\Pi_{\alpha_0} = \{(\alpha, x) : \alpha = \alpha_0\}$ are also invariant with respect to (5.9). Therefore, the manifold $\mathcal{W}^c$ is foliated by n_0-dimensional invariant manifolds

$$W^c_\alpha = \mathcal{W}^c \cap \Pi_\alpha$$

by n_0-dimensional invariant manifolds

$$W^c_\alpha = \mathcal{W}^c \cap \Pi_\alpha$$

(see Figure 5.7). Thus, we have the following lemma.

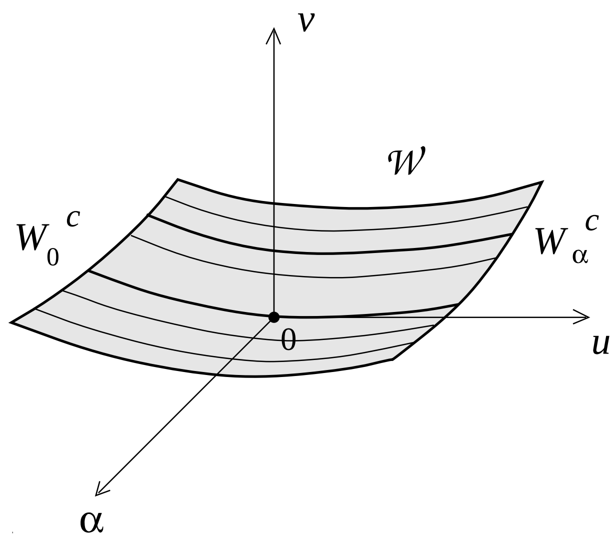


FIGURE 5.7. Center manifold of the extended system.

Lemma 5.1 *System* (5.8) *has a parameter-dependent local invariant manifold W^c_α. If $n_+ = 0$, this manifold is attracting.* □

Notice that W^c_0 is a center manifold of (5.9) at $\alpha = 0$ as defined in the previous section. Often, the manifold W^c_α is called a *center manifold* for all

α. For each small $|\alpha|$ we can restrict system (5.8) to W^c_α. If we introduce a (parameter-dependent) coordinate system on W^c_α with $u \in \mathbb{R}^{n_0}$ as the coordinate,[2] this restriction will be represented by a smooth system:

$$\dot{u} = \Phi(u, \alpha). \tag{5.10}$$

At $\alpha = 0$, system (5.10) is equivalent to the restriction of (5.8) to its center manifold W^c_0 and will be explicitly computed up to the third-order terms in Section 5.4 for all codim 1 bifurcations.

Theorem 5.4 (Shoshitaishvili [1975]) *System* (5.8) *is locally topologically equivalent to the suspension of* (5.10) *by the standard saddle* (5.4). *Moreover,* (5.10) *can be replaced by any locally topologically equivalent system.* □

This theorem means that all "essential" events near the bifurcation parameter value occur on the invariant manifold W^c_α and are captured by the n_0-dimensional system (5.10). A similar theorem can be formulated for discrete-time dynamical systems and for systems with more than one parameter. Let us apply this theorem to the fold and Hopf bifurcations.

Example 5.1 (Generic fold bifurcation in $\mathbb{R}^2$) Consider a planar system

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1. \tag{5.11}$$

Assume that at $\alpha = 0$ it has the equilibrium $x_0 = 0$ with one eigenvalue system

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1. \tag{5.11}$$

Assume that at $\alpha = 0$ it has the equilibrium $x_0 = 0$ with one eigenvalue $\lambda_1 = 0$ and one eigenvalue $\lambda_2 < 0$. Lemma 5.1 gives the existence of a smooth, locally defined, one-dimensional attracting invariant manifold W^c_α for (5.11) for small $|\alpha|$. At $\alpha = 0$ the restricted equation (5.10) has the form

$$\dot{u} = au^2 + O(u^3).$$

If $a \neq 0$ and the restricted equation depends generically on the parameter, then, as proved in Chapter 3, it is locally topologically equivalent to the normal form

$$\dot{u} = \alpha + \sigma u^2,$$

where $\sigma = \text{sign } a = \pm 1$. Under these genericity conditions, Theorem 5.4 implies that (5.11) is locally topologically equivalent to the system

$$\begin{cases} \dot{u} &= \alpha + \sigma u^2, \\ \dot{v} &= -v. \end{cases} \tag{5.12}$$

Equations (5.12) are decoupled. The resulting phase portraits are presented in Figure 5.8 for the case $\sigma > 0$. For $\alpha < 0$, there are two hyperbolic

---

[2] Since W^c_0 is tangent to T^c, we can parametrize W^c_α for small $|\alpha|$ by coordinates on T^c using a (local) projection from W^c_α onto T^c.

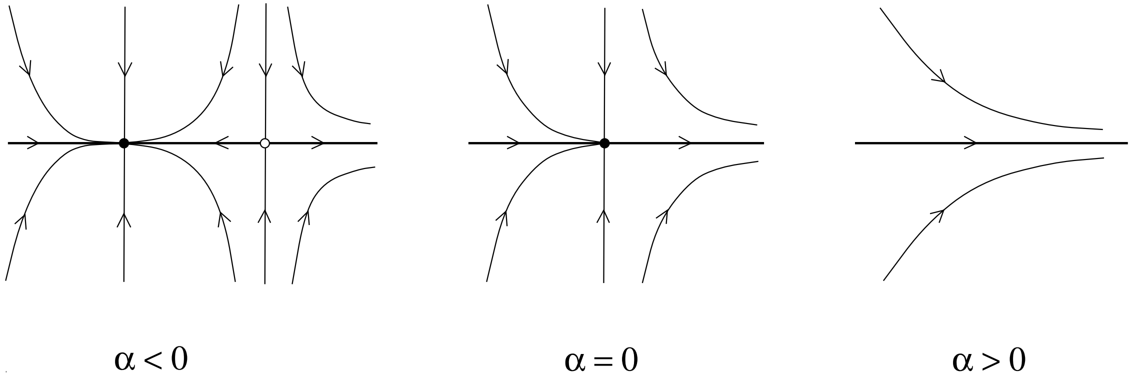
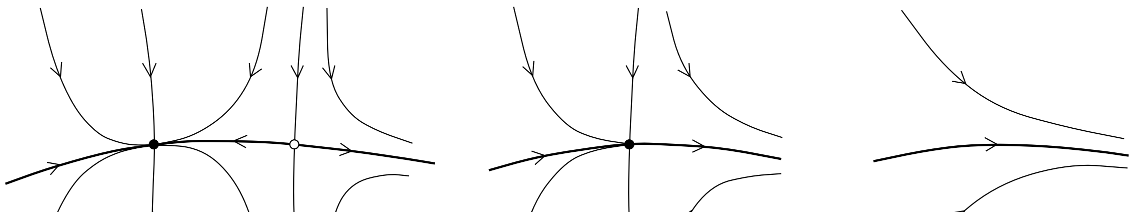


FIGURE 5.8. Fold bifurcation in the standard system (5.12) for $\sigma = 1$.

equilibria: a stable node and a saddle. They collide at $\alpha = 0$, forming a nonhyperbolic *saddle-node* point, and disappear. There are no equilibria for $\alpha > 0$. The manifolds W^c_α in (5.12) can be considered as parameter-independent and as given by $v = 0$. Obviously, it is one of the infinite number of choices (see the Remark following Example 5.2). The same events happen in (5.11) on some one-dimensional, parameter-dependent, invariant manifold, that is locally attracting (see Figure 5.9). All the equilibria belong



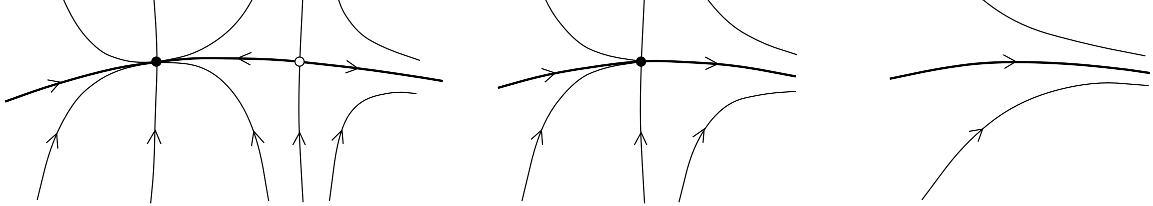


FIGURE 5.9. Fold bifurcation in a generic planar system.

to this manifold. Figures 5.8 and 5.9 explain why the fold bifurcation is often called the *saddle-node bifurcation*. It should be clear how to generalize these considerations to cover the case $\lambda_2 > 0$, as well as the n-dimensional case. ◇

Example 5.2 (Generic Hopf bifurcation in $\mathbb{R}^3$) Consider a system

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^3, \ \alpha \in \mathbb{R}^1. \tag{5.13}$$

Assume that at $\alpha = 0$ it has the equilibrium $x_0 = 0$ with eigenvalues $\lambda_{1,2} = \pm i\omega_0, \ \omega_0 > 0$ and one negative eigenvalue $\lambda_3 < 0$. Lemma 5.1 gives the existence of a parameter-dependent, smooth, local two-dimensional attracting invariant manifold W^c_α of (5.15) for small $|\alpha|$. At $\alpha = 0$ the restricted equation (5.10) can be written in complex form as

$$\dot{z} = i\omega_0 z + g(z, \bar{z}), \quad z \in \mathbb{C}^1.$$

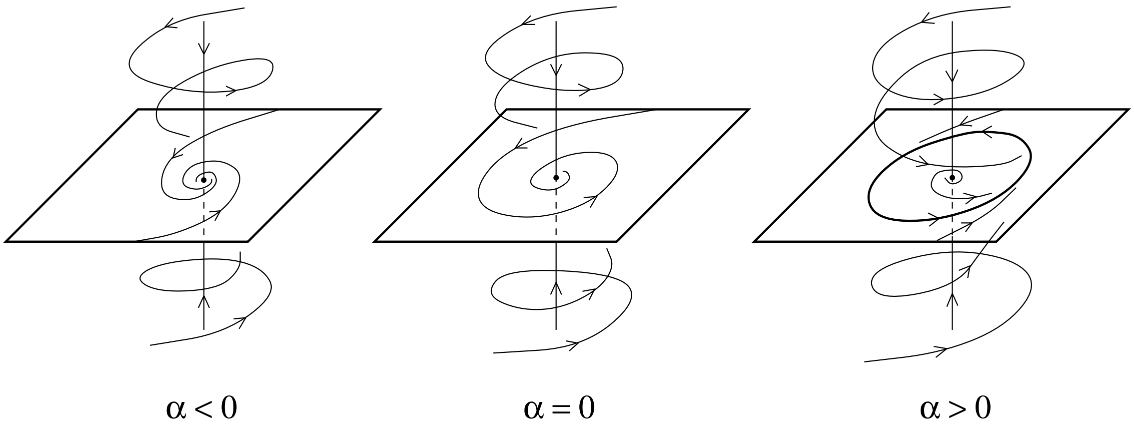


FIGURE 5.10. Hopf bifurcation in the standard system (5.14) for $\sigma = -1$.

If the Lyapunov coefficient $l_1(0)$ of this equation is nonzero and the restricted equation depends generically on the parameter, then, as proved in Chapter 3, it is locally topologically equivalent to the normal form

$$\dot{z} = (\alpha + i)z + \sigma z^2 \bar{z},$$

where $\sigma = \text{sign}\, l_1(0) = \pm 1$. Under these genericity conditions, Theorem 5.4 implies that (5.13) is locally topologically equivalent to the system

$$\begin{cases} \dot{z} &= (\alpha + i)z + \sigma z^2 \bar{z}, \\ \dot{v} &= -v. \end{cases} \tag{5.14}$$

The phase portrait of (5.14) is shown in Figure 5.8 for $\sigma = -1$. The su-

$$\left\{ \quad \dot{v} \quad = \quad -v. \right. \tag{5.14}$$

The phase portrait of (5.14) is shown in Figure 5.8 for $\sigma = -1$. The supercritical Hopf bifurcation takes place in the invariant plane $v = 0$, which is attracting. The same events happen for (5.13) on some two-dimensional attracting manifold (see Figure 5.11). The construction allows a generalization to arbitrary dimension $n \geq 3$. $\diamond$

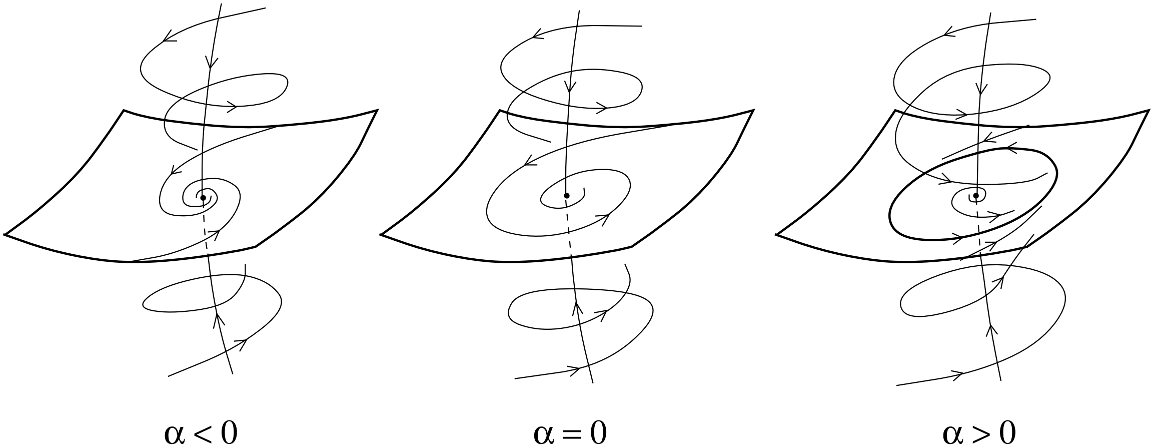


FIGURE 5.11. Supercritical Hopf bifurcation in a generic three-dimensional system.

Remark:

It should be noted that the manifold W^c_α is *not unique* in either the fold or Hopf cases, but the bifurcating equilibria or cycle belong to any of the center manifolds (cf. Remark (2) after the Center Manifold Theorem in Section 5.1.1). In the fold bifurcation case, the manifold is unique near the saddle and coincides with its unstable manifold as far as it exists. The uniqueness is lost at the stable node. Similarly, in the Hopf bifurcation case, the manifold is unique and coincides with the unstable manifold of the saddle-focus until the stable limit cycle L_α, where the uniqueness breaks down. Figure 5.12 shows the possible freedom in selecting W^c_α in the Hopf case for $\alpha > 0$ in (ρ, v)-coordinates in system (5.14) with $\sigma = -1$. ♢

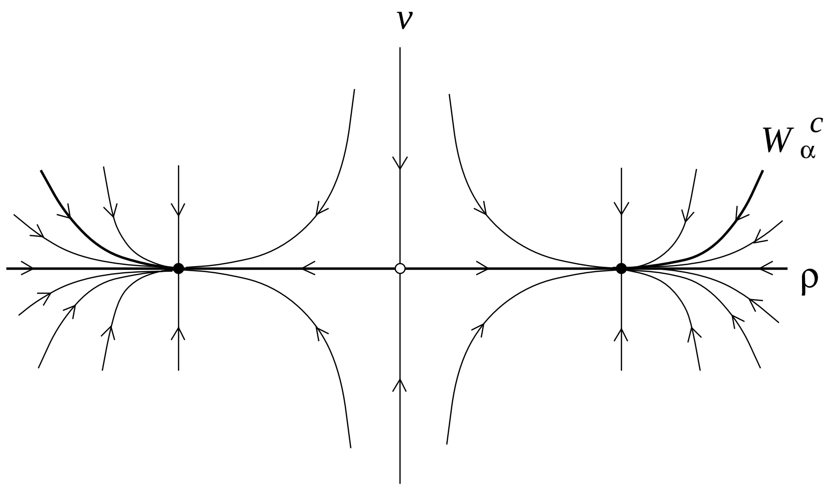


FIGURE 5.12. Nonuniqueness of the parameter-dependent center manifold near the Hopf bifurcation.

FIGURE 5.12. Nonuniqueness of the parameter-dependent center manifold near the Hopf bifurcation.

5.3 Bifurcations of limit cycles

A combination of the Poincaré map (see Chapter 1) and the center manifold approaches allows us to apply the results of Chapter 4 to limit cycle bifurcations in n-dimensional continuous-time systems.

Let L_0 be a limit cycle (isolated periodic orbit) of system (5.8) at $\alpha = 0$. Let P_α denote the associated Poincaré map for nearby α; $P_\alpha : \Sigma \to \Sigma$, where Σ is a local cross-section to L_0. If some coordinates $\xi = (\xi_1, \xi_2, \ldots, \xi_{n-1})$ are introduced on Σ, then $\tilde{\xi} = P_\alpha(\xi)$ can be defined to be the point of the next intersection with Σ of the orbit of (5.8) having initial point with coordinates ξ on Σ. The intersection of Σ and L_0 gives a fixed point ξ_0 for P_0: $P_0(\xi_0) = \xi_0$. The map P_α is smooth and locally invertible.

Suppose that the cycle L_0 is nonhyperbolic, having n_0 multipliers on the unit circle. The center manifold theorems then give a parameter-dependent invariant manifold $W^c_\alpha \subset \Sigma$ of P_α on which the "essential" events take place. The Poincaré map P_α is locally topologically equivalent to the suspension of its restriction to this manifold by the standard saddle map. Fix $n = 3$, for simplicity, and consider the implications of this theorem for the limit cycle bifurcations.

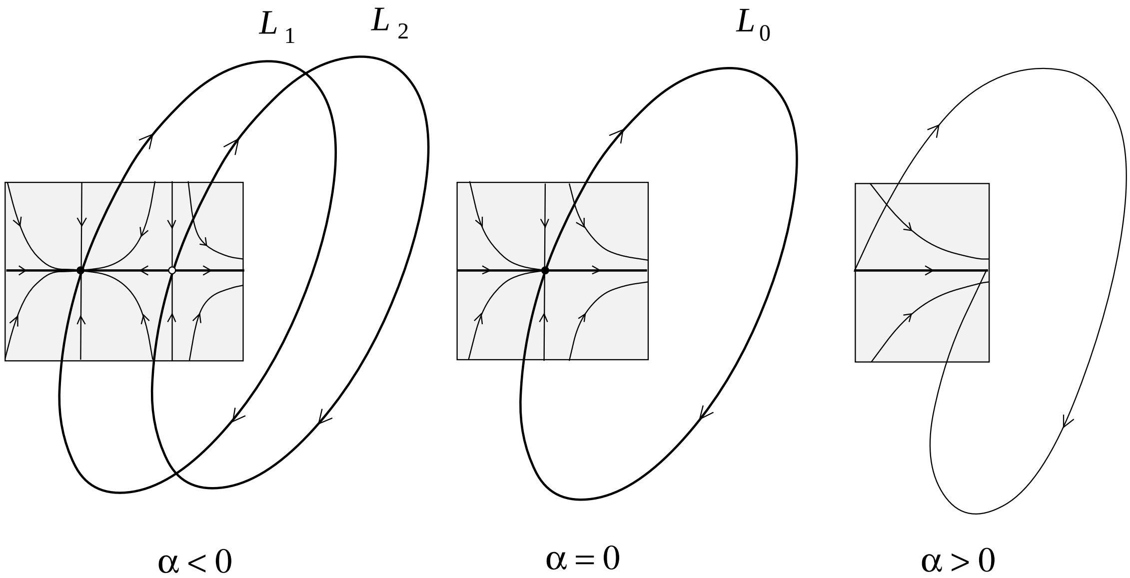


FIGURE 5.13. Fold bifurcation of limit cycles.

Fold bifurcation of cycles

Assume that at $\alpha = 0$ the cycle has a simple multiplier $\mu_1 = 1$ and its other multiplier satisfies $0 < \mu_2 < 1$. The restriction of P_α to the invariant manifold W_α^c is a one-dimensional map, having a fixed point with $\mu_1 = 1$ at $\alpha = 0$. As has been shown in Chapter 4, this generically implies the collision and disappearance of two fixed points of P_α as α passes through zero. Under our assumption on μ_2, this happens on a one-dimensional attracting

$\alpha = 0$. As has been shown in Chapter 4, this generically implies the collision and disappearance of two fixed points of P_α as α passes through zero. Under our assumption on μ_2, this happens on a one-dimensional attracting invariant manifold of P_α; thus, a stable and a saddle fixed point are involved in the bifurcation (see Figure 5.13). Each fixed point of the Poincaré map corresponds to a limit cycle of the continuous-time system. Therefore, two limit cycles (stable and saddle) collide and disappear in system (5.8) at this bifurcation (see the figure).

Flip bifurcation of cycles

Suppose that at $\alpha = 0$ the cycle has a simple multiplier $\mu_1 = -1$, while $-1 < \mu_2 < 0$. Then, the restriction of P_α to the invariant manifold will demonstrate generically the period-doubling (flip) bifurcation: A cycle of period two appears for the map, while the fixed point changes its stability (see Figure 5.14). Since the manifold is attracting, the stable fixed point, for example, loses stability and becomes a saddle point, while a stable cycle of period two appears. The fixed points correspond to limit cycles of the relevant stability. The cycle of period-two points for the map corresponds to a unique stable limit cycle in (5.8) with *approximately* twice the period of the "basic" cycle L_0. The double-period cycle makes two big "excursions" near L_0 before the closure. The exact bifurcation scenario is determined by the normal form coefficient of the restricted Poincaré map evaluated at $\alpha = 0$.

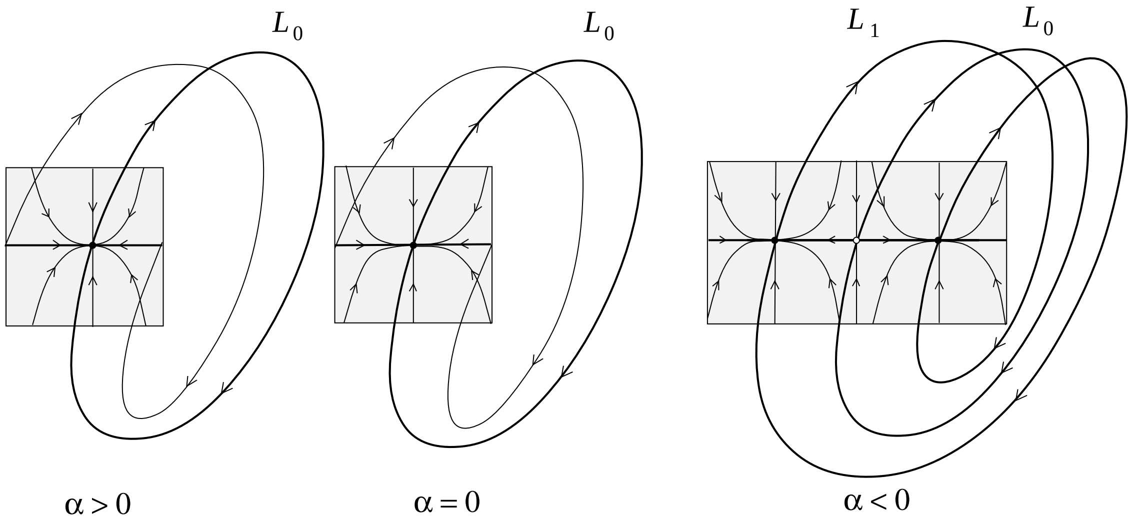


FIGURE 5.14. Flip bifurcation of limit cycles.

Neimark-Sacker bifurcation of cycles

The last codim 1 bifurcation corresponds to the case when the multipliers are complex and simple and lie on the unit circle: $\mu_{1,2} = e^{\pm i\theta_0}$. The Poincaré map then has a parameter-dependent, two-dimensional, invariant manifold on which a closed invariant curve generically bifurcates from the fixed point (see Figure 5.15). This closed curve corresponds to a two-dimensional *invariant torus* $\mathbb{T}^2$ in (5.8). The bifurcation is determined by the normal form coefficient of the restricted Poincaré map at the critical the fixed point (see Figure 5.15). This closed curve corresponds to a two-dimensional *invariant torus* $\mathbb{T}^2$ in (5.8). The bifurcation is determined by the normal form coefficient of the restricted Poincaré map at the critical parameter value. The orbit structure on the torus $\mathbb{T}^2$ is determined by the restriction of the Poincaré map to this closed invariant curve. Thus, generically, there are long-period cycles of different stability types located on the torus (see Chapter 7).

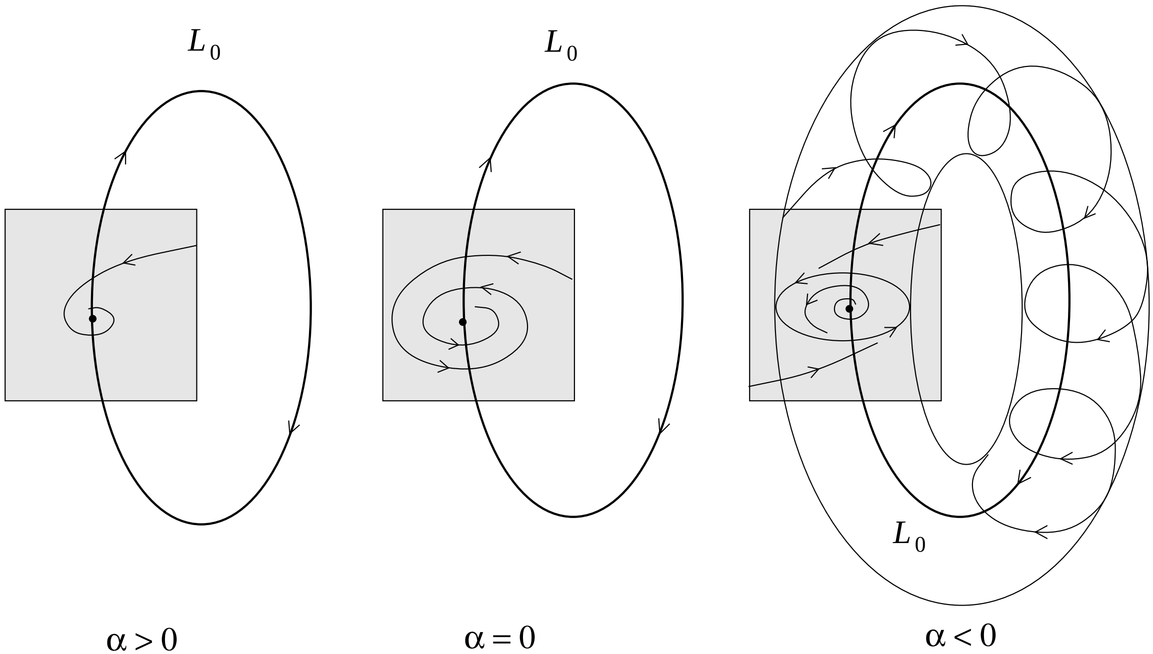


FIGURE 5.15. Neimark-Sacker bifurcation of a limit cycle.

5.4 Computation of center manifolds

As pointed out in the previous sections, the analysis of bifurcations of equilibria and fixed points (and, therefore, limit cycles) in multidimensional systems reduces to that for the equations (maps) restricted to the invariant manifold W^c_α. Since these bifurcations are determined by the normal form coefficients of the restricted systems at the critical parameter value $\alpha = 0$, we have to be able to compute the center manifold $W^c = W^c_0$ and the equations or maps restricted to this manifold up to sufficiently high-order terms. Coefficients of the Taylor expansion of the function $v = V(u)$ representing the center manifold W^c can be computed via a recursive procedure, each step of which involves solving a linear system of algebraic equations. The coefficients so obtained are the same for all nonunique center manifolds of the system. In the C^∞ case this means that these manifolds can only differ by "flat" functions. Ahead, we derive explicit formulas for the quadratic Taylor coefficients of the center manifolds for all codim 1 bifurcations of equilibria and fixed points. As should now be clear, for these cases W^c is either one- or two-dimensional, $n_0 = 1, 2$. Note that in order to analyze these bifurcations it is sufficient, in the generic case, to obtain the restricted equations up to (and including) third-order terms only.

5.4.1 Quadratic approximation to center manifolds in

5.4.1 *Quadratic approximation to center manifolds in eigenbasis*

In this section we assume that the system at the bifurcation parameter value is transformed into its eigenbasis and has the form (5.2) or (5.6). In the next section we will show how to avoid this transformation while leaving the obtained formulas virtually unchanged. Thus, in practice, this latter method should be used in the analysis of systems arising in applications, since they are almost never written in the eigenform (5.2) or (5.6).

Let us start with the continuous-time systems.

Fold bifurcation ($\lambda_1 = 0$)

In this case, $n_0 = 1$ and system (5.2) can be written as

$$\left\{ \begin{array}{rcl} \dot{u} & = & \frac{1}{2}\sigma u^2 + u\langle b, v\rangle + \frac{1}{6}\delta u^3 + \cdots, \\ \dot{v} & = & Cv + \frac{1}{2}au^2 + \cdots, \end{array} \right. \tag{5.15}$$

where $u \in \mathbb{R}^1$, $v \in \mathbb{R}^{n-1}$, $\sigma, \delta \in \mathbb{R}^1$, $a, b \in \mathbb{R}^{n-1}$, and C is an $(n-1)\times(n-1)$ matrix without eigenvalues on the imaginary axis. Here $\langle b, v\rangle = \sum_{i=1}^{n-1} b_i v_i$ is the standard scalar product in $\mathbb{R}^{n-1}$, and the dots mean all undisplayed terms.[3] Using the functions h and g from (5.2), we obtain

$$\sigma = \frac{\partial^2}{\partial u^2} g(u, 0)\bigg|_{u=0}, \tag{5.16}$$

$$\delta = \frac{\partial^3}{\partial u^3} g(u, 0)\bigg|_{u=0}, \tag{5.17}$$

$$a = \frac{\partial^2}{\partial u^2} h(u, 0)\bigg|_{u=0}, \tag{5.18}$$

and the components $(b_1, b_2, \ldots, b_{n-1})$ of the vector b are given by

$$b_i = \frac{\partial^2}{\partial v_i \partial u} g(u, v)\bigg|_{u=0, v=0}, \tag{5.19}$$

where $i = 1, 2, \ldots, n-1$.

We seek the second-order term in the Taylor expansion for $v = V(u)$ representing the center manifold:

$$v = V(u) = \tfrac{1}{2} w_2 u^2 + O(u^3), \tag{5.20}$$

where $w_2 \in \mathbb{R}^{n-1}$ is an unknown vector. Substituting expansion (5.20) into the second equation of (5.15), and using the first equation, we get

$$w_2 u(\sigma u^2 + \langle b, w_2 u^2 \rangle) + O(u^4) = C w_2 u^2 + a u^2 + O(u^3)$$

where $w_2 \in \mathbb{R}^{n-1}$ is an unknown vector. Substituting expansion (5.20) into the second equation of (5.15), and using the first equation, we get

$$w_2 u(\sigma u^2 + \langle b, w_2 u^2 \rangle) + O(u^4) = C w_2 u^2 + a u^2 + O(u^3),$$

which results in the following linear equation for w_2 at u^2-terms:

$$C w_2 + a = 0.$$

This linear system has a unique solution, since C is invertible (because $\lambda = 0$ is not an eigenvalue of C). Thus,

$$w_2 = -C^{-1} a,$$

and the restriction of (5.15) to the center manifold (5.22) up to (and including) the third-order term is given by

$$\dot{u} = \tfrac{1}{2}\sigma u^2 + \tfrac{1}{6}\left(\delta - 3\langle b, C^{-1} a \rangle\right) u^3 + O(u^4). \tag{5.21}$$

Notice that, in fact, the quadratic term in (5.21) is exactly the same as in the first equation of (5.15). Thus, to analyze the fold (tangent) bifurcation, the linear approximation to the center manifold is sufficient, provided $\sigma \neq$

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[3] For example, $O(\|v\|^2)$ terms in both equations of (5.15), and $O(|u|\|v\|)$ terms in the second equation of (5.15) are irrelevant in the following, because they do not affect the quadratic terms of the restricted equations.

0. It is enough, therefore, to substitute $v = 0$ into the first equation of (5.15). This way of approximating W^c by the eigenspace T^c obviously fails to determine even stability of the equilibrium if $\sigma = 0$.

Example 5.3 (Failure of the tangent approximation) Consider the following planar system:

$$\begin{cases} \dot{x} &=& xy + x^3, \\ \dot{y} &=& -y - 2x^2. \end{cases} \tag{5.22}$$

There is an equilibrium at $(x, y) = (0, 0)$. Is it stable or unstable? The Jacobian matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$. Thus, system (5.22) is written in the form (5.2) and has a one-dimensional center manifold W^c represented by the scalar function

$$y = V(x) = \tfrac{1}{2}wx^2 + \cdots.$$

Then,

$$\dot{y} = wx\dot{x} + \cdots = wx^2y + wx^4 + \cdots = w\left(\tfrac{1}{2}w + 1\right)x^4 + \cdots,$$

or alternatively,

$$\dot{y} = wx\dot{x} + \cdots = wx^2y + wx^4 + \cdots = w\left(\tfrac{1}{2}w + 1\right)x^4 + \cdots,$$

or alternatively,

$$\dot{y} = -y - 2x^2 = -\left(\tfrac{1}{2}w + 2\right)x^2 + \cdots.$$

Therefore, $w + 4 = 0$ and

$$w = -4.$$

Thus, the center manifold has the following quadratic approximation:

$$V(x) = -2x^2 + O(x^3),$$

and the restriction of (5.22) to its center manifold is given by

$$\dot{x} = xV(x) + x^3 = -2x^3 + x^3 + O(x^4) = -x^3 + O(x^4).$$

Therefore, the origin is *stable* and the phase portrait of the system near the equilibrium is as sketched in Figure 5.16. By restriction of (5.22) onto its critical eigenspace $y = 0$, one gets

$$\dot{x} = x^3.$$

This equation has an *unstable* point at the origin and thus gives the wrong answer to the stability question. Figure 5.17 compares the equations restricted to W^c and T^c. $\diamond$

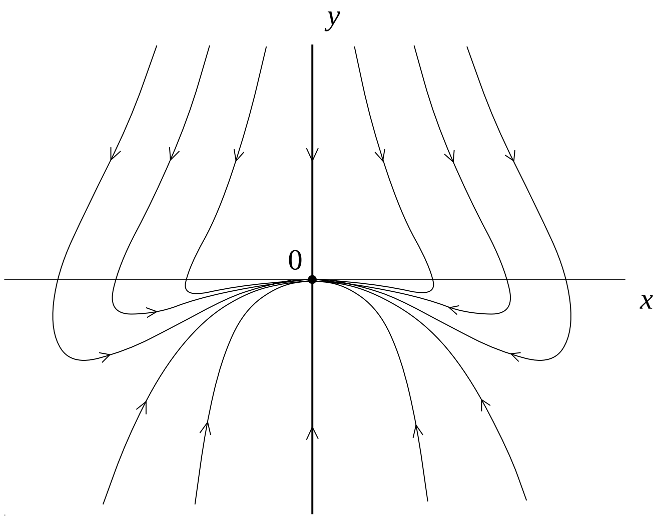
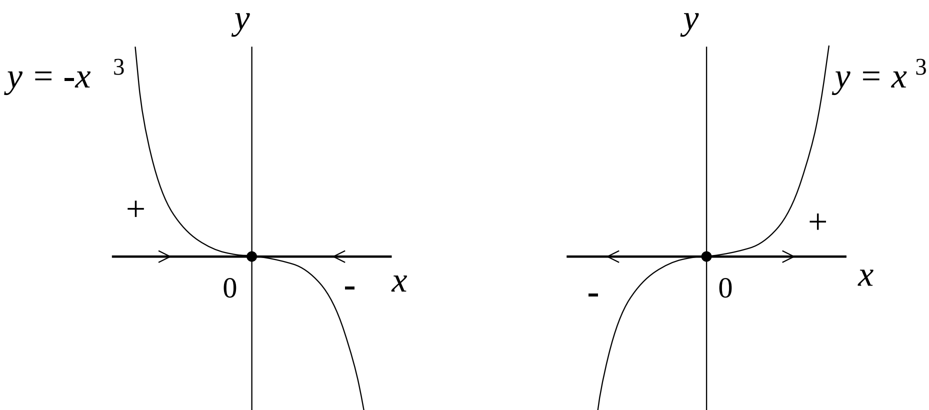
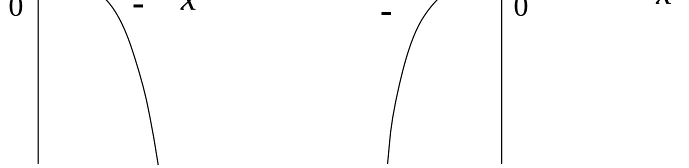


FIGURE 5.16. Phase portrait of (5.22): The origin is stable.





(a) (b)

FIGURE 5.17. Restricted equations: (a) to the center manifold W^c; (b) to the tangent line T^c.

Hopf bifurcation ($\lambda_{1,2} = \pm i\omega_0$)

Now $n_0 = 2$ and system (5.1) in its eigenbasis takes the form

$$\begin{cases} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} G_1(u_1, u_2, v) \\ G_2(u_1, u_2, v) \end{pmatrix}, \\ \dot{v} = Cv + H_1(u_1, u_2, v), \end{cases} \tag{5.23}$$

where $u = (u_1, u_2)^T \in \mathbb{R}^2,\ v \in \mathbb{R}^{n-2}$. It is convenient to rewrite (5.23) into *complex* form by introducing $z = u_1 + iu_2$:

$$\begin{cases} \dot{z} = i\omega_0 z + G(z, \bar{z}, v), \\ \dot{v} = Cv + H(z, \bar{z}, v). \end{cases} \tag{5.24}$$

Here G and H are smooth complex-valued functions of $z, \bar{z} \in \mathbb{C}^1$, and $v \in \mathbb{R}^{n-2}$. Actually, z can be viewed as a new "coordinate" on the critical eigenspace $T^c = \{v = 0\}$ of (5.23). The center manifold W^c therefore has the representation

$$v = V(z, \bar{z}) = \frac{1}{2} w_{20} z^2 + w_{11} z\bar{z} + \frac{1}{2} w_{02} \bar{z}^2 + O(|z|^3), \tag{5.25}$$

with unknown $w_{ij} \in \mathbb{C}^{n-2}$. Since V must be real, w_{11} is real and $w_{20} = \bar{w}_{02}$.

Let us write system (5.24) in more detail using the Taylor expansions in $z, \bar{z}$, and v:

$$\begin{cases} \dot{z} &= i\omega_0 z + \frac{1}{2} G_{20} z^2 + G_{11} z\bar{z} + \frac{1}{2} G_{02} \bar{z}^2 + \frac{1}{2} G_{21} z^2 \bar{z} \\ & \quad + \langle G_{10}, v \rangle z + \langle G_{01}, v \rangle \bar{z} + \cdots, \\ \dot{v} &= Cv + \frac{1}{2} H_{20} z^2 + H_{11} z\bar{z} + \frac{1}{2} H_{02} \bar{z}^2 + \cdots, \end{cases} \tag{5.26}$$

where $G_{20}, G_{11}, G_{02}, G_{21} \in \mathbb{C}^1$; $G_{01}, G_{10}, H_{ij} \in \mathbb{C}^{n-2}$; H_{11} is real; and $H_{20} = \overline{H}_{02}$. The scalar product now means that $\langle G, v \rangle = \sum_{i=1}^{n-2} \overline{G}_i v_i$. In terms of the functions G and H from (5.24), we get

$$G_{ij} = \left. \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} G(z, \bar{z}, 0) \right|_{z=0}, \quad i + j \geq 2, \tag{5.27}$$

$$\overline{G}_{10,i} = \left. \frac{\partial^2}{\partial v_i \partial z} G(z, \bar{z}, v) \right|_{z=0, v=0}, \quad i = 1, 2, \ldots, n-2, \tag{5.28}$$

$$\overline{G}_{01,i} = \left. \frac{\partial^2}{\partial v_i \partial \bar{z}} G(z, \bar{z}, v) \right|_{z=0, v=0}, \quad i = 1, 2, \ldots, n-2, \tag{5.29}$$

$$\overline{G}_{01,i} = \left.\frac{\partial^2}{\partial v_i \partial \bar{z}} G(z, \bar{z}, v)\right|_{z=0, v=0}, \quad i = 1, 2, \ldots, n-2, \tag{5.29}$$

$$H_{ij} = \left.\frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} H(z, \bar{z}, 0)\right|_{z=0}, \quad i+j = 2. \tag{5.30}$$

Substitution of (5.25) into (5.26) gives, at the quadratic level,

$$\begin{cases} (2i\omega_0 E - C)w_{20} = H_{20}, \\ -Cw_{11} = H_{11}, \\ (-2i\omega_0 E - C)w_{02} = H_{02}. \end{cases} \tag{5.31}$$

Thus,

$$\begin{aligned} w_{20} &= (2i\omega_0 E - C)^{-1} H_{20}, \\ w_{11} &= -C^{-1} H_{11}, \\ w_{02} &= (-2i\omega_0 E - C)^{-1} H_{02}. \end{aligned}$$

Here E is the identity matrix and the matrices $(2i\omega_0 E - C), C, (-2i\omega_0 E - C)$ are invertible, since 0 and $\pm 2i\omega_0$ are *not* eigenvalues of C. Now, the restriction of (5.24) to its center manifold, up to cubic terms, can be written as follows:

$$\begin{aligned} \dot{z} &= i\omega_0 z + \tfrac{1}{2} G_{20} z^2 + G_{11} z\bar{z} + \tfrac{1}{2} G_{02} \bar{z}^2 \\ &+ \tfrac{1}{2}(G_{21} - 2\langle G_{10}, C^{-1} H_{11}\rangle + \langle G_{01}, (2i\omega_0 E - C)^{-1} H_{20}\rangle) z^2 \bar{z} + \cdots, \end{aligned} \tag{5.32}$$

where only the cubic term needed for the Hopf bifurcation analysis is displayed.

Now consider the discrete-time case.

Fold bifurcation of maps ($\mu_1 = 1$)

In this case, system (5.6) can be written as

$$\begin{cases} \tilde{u} &= u + \frac{1}{2}\sigma u^2 + u\langle b, v\rangle + \frac{1}{6}\delta u^3 + \cdots, \\ \tilde{v} &= Cv + \frac{1}{2}au^2 + \cdots, \end{cases} \tag{5.33}$$

where $u, \tilde{u} \in \mathbb{R}^1,\ v, \tilde{v} \in \mathbb{R}^{n-1};\ \sigma, \delta \in \mathbb{R}^1,\ a, b \in \mathbb{R}^{n-1}$ are given by equations (5.16) through (5.19); and C is an $(n-1) \times (n-1)$ matrix without eigenvalues on the unit circle. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{n-1}$, and only the terms needed in what follows are presented. The center manifold of (5.33) is given by

$$v = V(u) = \frac{1}{2}w_2 u^2 + O(u^3),$$

where $w_2 \in \mathbb{R}^{n-1}$ is unknown. Substituting this expansion into the second equation of (5.33), using the first equation and the *invariance* of the center manifold (if $v = V(u)$, then $\tilde{v} = V(\tilde{u})$), we get the following linear equation for w_2, collecting u^2-terms:

$$(C - E)w_2 + a = 0. \tag{5.34}$$

for w_2, collecting u^2-terms:

$$(C - E)w_2 + a = 0. \tag{5.34}$$

This linear system has a unique solution, since $(C-E)$ is invertible because $\mu = 1$ is not an eigenvalue of C. Thus,

$$w_2 = (E - C)^{-1}a,$$

and the restriction of (5.33) to the center manifold, up to (and including) the third-order term, is given by

$$u \mapsto u + \tfrac{1}{2}\sigma u^2 + \tfrac{1}{6}\left(\delta + 3\langle b, (E - C)^{-1}a\rangle\right) u^3 + O(u^4). \tag{5.35}$$

The quadratic term in (5.35) is exactly the same as in the first equation of (5.33). Thus, to analyze the fold (tangent) bifurcation of maps we need no nonlinear approximations to the center manifold, provided $\sigma \neq 0$. Substitution of $v = 0$ into the first equation of (5.33) gives the correct restriction up to second-order terms.

Flip bifurcation ($\mu_1 = -1$)

Now system (5.6) can be written as

$$\left\{ \begin{array}{rcl} \tilde{u} & = & -u + \frac{1}{2}\sigma u^2 + u\langle b, v\rangle + \frac{1}{6}\delta u^3 + \cdots, \\ \tilde{v} & = & Cv + \frac{1}{2}au^2 + \cdots, \end{array} \right. \tag{5.36}$$

with the same notation as in the previous case, and σ, δ, a, and b are given by equations (5.16)–(5.19). The center manifold is again represented by $v = V(u) = \frac{1}{2}w_2u^2 + O(u^3)$, where $w_2 \in \mathbb{R}^{n-1}$ is a vector satisfying the same linear equation (5.34). Therefore, the restriction of (5.36) to the center manifold is

$$u \mapsto -u + \frac{1}{2}\sigma u^2 + \frac{1}{6}\left(\delta + 3\langle b, (E - C)^{-1}a\rangle\right) u^3 + O(u^4).$$

Neimark-Sacker bifurcation ($\mu_{1,2} = e^{\pm i\theta_0}$)

In the eigenbasis and written with complex notation, system (5.6) can be denoted as

$$\left\{\begin{array}{rcl} \tilde{z} & = & e^{i\theta_0} z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z} \\ & & + \langle G_{10}, v\rangle z + \langle G_{01}, v\rangle \bar{z} + \cdots, \\ \tilde{v} & = & Cv + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \cdots, \end{array}\right. \tag{5.37}$$

where the notation is the same as in the Hopf case, and G_{ij} and H_{ij} are given by the expressions (5.27)–(5.30). The center manifold of (5.37) has the representation

$$v = V(z, \bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2 + O(|z|^3),$$

with $w_{ij} \in \mathbb{C}^{n-2}$. The Taylor coefficients satisfy the linear equations

$$v = V(z, \bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2 + O(|z|^3),$$

with $w_{ij} \in \mathbb{C}^{n-2}$. The Taylor coefficients satisfy the linear equations

$$\left\{ \begin{array}{rcl} (e^{2i\theta_0}E - C)w_{20} & = & H_{20}, \\ (E - C)w_{11} & = & H_{11}, \\ (e^{-2i\theta_0}E - C)w_{02} & = & H_{02}. \end{array} \right. \tag{5.38}$$

Thus,

$$\begin{array}{rcl} w_{20} & = & (e^{2i\theta_0}E - C)^{-1}H_{20}, \\ w_{11} & = & (E - C)^{-1}H_{11}, \\ w_{02} & = & (e^{-2i\theta_0}E - C)^{-1}H_{02}. \end{array}$$

The matrices in (5.38) are invertible since $e^{\pm 2i\theta_0}$ and 1 are not eigenvalues of C. The restriction of (5.37) to the center manifold therefore has the form

$$\begin{array}{rcl} z & \mapsto & e^{i\theta_0}z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z} \\ & + & \frac{1}{2}\left(2\langle G_{10}, (E - C)^{-1}H_{11}\rangle + \langle G_{01}, (e^{2i\theta_0}E - C)^{-1}H_{20}\rangle\right) z^2\bar{z} \ + \cdots, \end{array}$$

where the only cubic terms retained are those that are necessary for analyzing a generic Neimark-Sacker bifurcation.

5.4.2 Projection method for center manifold computation

There is a useful method for center manifold computation which avoids the transformation of the system into its eigenbasis (to the form (5.2) or

(5.6)). Instead, only eigenvectors corresponding to the critical eigenvalues of A and its transpose A^T are used to "project" the system into the critical eigenspace and its complement. This method can be applied to both continuous- and discrete-time finite-dimensional systems, as well as to some infinite-dimensional systems (see Appendix 1) with few modifications.

As usual, we start with the continuous-time case. Suppose system (5.1) is written as

$$\dot{x} = Ax + F(x), \quad x \in \mathbb{R}^n, \tag{5.39}$$

where $F(x) = O(\|x\|^2)$ is a smooth function.

Fold bifurcation

In this case, A has a simple zero eigenvalue $\lambda_1 = 0$, and the corresponding critical eigenspace T^c is one-dimensional and spanned by an eigenvector $q \in \mathbb{R}^n$ such that $Aq = 0$. Let $p \in \mathbb{R}^n$ be the *adjoint* eigenvector, that is, $A^T p = 0$, where A^T is the transposed matrix.[4] It is possible and convenient to normalize p with respect to q : $\langle p, q \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. The following lemma follows from the Fredholm Alternative Theorem.

Lemma 5.2 *Let T^{su} denote an $(n-1)$-dimensional linear eigenspace of A corresponding to all eigenvalues other than 0. Then $y \in T^{su}$ if and only if $\langle p, y \rangle = 0$.* □

Lemma 5.2 *Let T^{su} denote an $(n-1)$-dimensional linear eigenspace of A corresponding to all eigenvalues other than 0. Then $y \in T^{su}$ if and only if $\langle p, y \rangle = 0$.* $\square$

Using the lemma, we can "decompose" any vector $x \in \mathbb{R}^n$ as

$$x = uq + y,$$

where $uq \in T^c$, $y \in T^{su}$. If q and p are normalized as above, we get explicit expressions for u and y:

$$\left\{ \begin{array}{rcl} u & = & \langle p, x \rangle, \\ y & = & x - \langle p, x \rangle q. \end{array} \right. \tag{5.40}$$

Two operators can thus be defined:

$$P_c x = \langle p, x \rangle q, \quad P_{su} x = x - \langle p, x \rangle q.$$

These operators are *projections* onto T^c and T^{su}, respectively, and

$$P_c^2 = P_c, \ P_{su}^2 = P_{su}, \ P_c P_{su} = P_{su} P_c = 0.$$

The scalar u and the vector y can be considered as new "coordinates" on $\mathbb{R}^n$. Although $y \in \mathbb{R}^n$, its components always satisfy the orthogonality

[4] Recall that $\langle x, Ay \rangle = \langle A^T x, y \rangle$ for any $x, y \in \mathbb{R}^n$.

condition $\langle p, y\rangle = 0$. In these new coordinates, system (5.39) can be written as

$$\begin{cases} \dot{u} &= \langle p, F(uq+y)\rangle, \\ \dot{y} &= Ay + F(uq+y) - \langle p, F(uq+y)\rangle q. \end{cases} \tag{5.41}$$

To obtain these equations, one has to take into account (5.40) and the eigenvector definitions and normalizations. Equivalently, one can apply the above projection operators to system (5.39). Using Taylor expansions, we can write (5.41) in a form similar to (5.15):

$$\begin{cases} \dot{u} &= \frac{1}{2}\sigma u^2 + u\langle b, y\rangle + \frac{1}{6}\delta u^3 + \cdots, \\ \dot{y} &= Ay + \frac{1}{2}au^2 + \cdots, \end{cases} \tag{5.42}$$

where $u \in \mathbb{R}^1,\ y \in \mathbb{R}^n,\ \sigma, \delta \in \mathbb{R}^1,\ a, b \in \mathbb{R}^n$, and $\langle b, y\rangle = \sum_{i=1}^n b_i y_i$ is now the standard scalar product in $\mathbb{R}^n$. For σ, δ, a, and b we get the following expressions:

$$\sigma = \left. \frac{\partial^2}{\partial u^2}\langle p, F(uq)\rangle \right|_{u=0}, \tag{5.43}$$

$$\delta = \left. \frac{\partial^3}{\partial u^3}\langle p, F(uq)\rangle \right|_{u=0}, \tag{5.44}$$

$$a = \left. \frac{\partial^2}{\partial u^2} F(uq) \right|_{u=0} - \sigma q, \tag{5.45}$$

and the components of the vector b are given by

$$a = \left. \frac{\partial^2}{\partial u^2} F(uq) \right|_{u=0} - \sigma q, \tag{5.45}$$

and the components of the vector b are given by

$$b_i = \left. \frac{\partial^2}{\partial y_i \partial u} \langle p, F(uq + y) \rangle \right|_{u=0, y=0}, \tag{5.46}$$

where $i = 1, 2, \ldots, n$.

We can now proceed *exactly* in the same way as in Section 5.4.1. The center manifold has the representation

$$y = V(u) = \tfrac{1}{2} w_2 u^2 + O(u^3),$$

where now $w_2 \in T^{su} \subset \mathbb{R}^n$, which means $\langle p, w_2 \rangle = 0$. The vector w_2 satisfies an equation in $\mathbb{R}^n$ that formally resembles the corresponding equation in Section 5.4.1,

$$A w_2 + a = 0. \tag{5.47}$$

Here, however, we have a slight complication, since A is obviously noninvertible in $\mathbb{R}^n$ ($\lambda = 0$ is its eigenvalue). This difficulty is easy to overcome. Notice that $a \in T^{su}$, since $\langle p, a \rangle = 0$. The restriction of the linear transformation corresponding to A to its invariant subspace T^{su} is invertible. Thus, equation (5.47) has a unique solution $w_2 \in T^{su}$. If we denote this solution by

$$w_2 = -A^{INV} a,$$

the restriction of (5.42) to the center manifold takes the form

$$\dot{u} = \frac{1}{2}\sigma u^2 + \frac{1}{6}\left(\delta - 3\langle b, A^{INV}a\rangle\right) u^3 + O(u^4). \tag{5.48}$$

To check that a fold bifurcation is nondegenerate, we need only to compute σ. For this, the linear approximation of W^c is enough, and σ is given by (5.43), where f from (5.1) can be used instead of F. If $\sigma = 0$, the third-order term must be computed.

Actually, explicit computation of the vector b using (5.46) is not necessary for finding the restricted equation. Indeed, let the function $F(x)$ be written as

$$F(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(\|x\|^4),$$

where $B(x,y)$ and $C(x,y,z)$ are *multilinear* functions. In coordinates, we have

$$B_i(x,y) = \sum_{j,k=1}^{n} \left.\frac{\partial^2 F_i(\xi)}{\partial\xi_j \partial\xi_k}\right|_{\xi=0} x_j y_k,$$

and

$$C_i(x,y,z) = \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 F_i(\xi)}{\partial\xi_j \partial\xi_k \partial\xi_l}\right|_{\xi=0} x_j y_k z_l,$$

where $i = 1, 2, \ldots, n$. Then the scalar product $\langle b, y\rangle$ can be expressed as

$$\langle b, y\rangle = \langle p, B(q,y)\rangle,$$

where $i = 1, 2, \ldots, n$. Then the scalar product $\langle b, y \rangle$ can be expressed as

$$\langle b, y \rangle = \langle p, B(q, y) \rangle,$$

and the restricted equation (5.48) takes the form

$$\dot{u} = \tfrac{1}{2}\sigma u^2 + \tfrac{1}{6}\left(\delta - 3\langle p, B(q, A^{INV} a)\rangle\right) u^3 + O(u^4), \tag{5.49}$$

where

$$\sigma = \langle p, B(q, q) \rangle, \ \ \delta = \langle p, C(q, q, q) \rangle, \ \ a = B(q, q) - \langle p, B(q, q) \rangle q. \tag{5.50}$$

Remarks:

(1) One can compute $w = A^{INV} a$ by solving the following $(n+1)$-dimensional *bordered system*

$$\begin{pmatrix} A & q \\ p^T & 0 \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{5.51}$$

for $w \in \mathbb{R}^n$ and $u \in \mathbb{R}^1$. Here q and p are the above-defined and normalized eigenvectors of A and A^T, respectively. The $(n+1) \times (n+1)$ matrix of this system is nonsingular. Indeed,

$$\begin{pmatrix} A & q \\ p^T & 0 \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

implies $w = 0$ and $u = 0$, so the null-space of the bordered matrix is trivial. Suppose now that $(w, u)^T$ is the solution to (5.51). Equivalently,

$$\begin{cases} Aw + uq = a, \\ \langle p, w \rangle = 0. \end{cases}$$

Thus, according to the second equation, $w \in T^{su}$. Taking the scalar product of the first equation with p, we obtain

$$\langle p, Aw \rangle + u \langle p, q \rangle = \langle p, a \rangle.$$

However, $\langle p, q \rangle = 1, \langle p, a \rangle = 0$, and $\langle p, Aw \rangle = \langle A^T p, w \rangle = 0$. Therefore, $u = 0$ and

$$Aw = a.$$

Thus, by definition, $w = A^{INV} a$.

(2) The choice of normalization for q is irrelevant. Indeed, if the vector q is substituted by γq with some nonzero $\gamma \in \mathbb{R}^1$ but the relative normalization $\langle p, q \rangle = 1$ is preserved, the coefficients of the restricted equation will change, although the equation can easily be scaled back to the original form by the substitution $u \mapsto \frac{1}{\gamma} u$. For the quadratic and cubic terms this can easily be seen from (5.49) and (5.50). ♢

Hopf bifurcation

seen from (5.49) and (5.50). $\diamond$

Hopf bifurcation

In this case, A has a simple pair of complex eigenvalues on the imaginary axis: $\lambda_{1,2} = \pm i\omega_0, \ \omega_0 > 0$, and these eigenvalues are the only eigenvalues with Re $\lambda = 0$. Let $q \in \mathbb{C}^n$ be a *complex* eigenvector corresponding to λ_1:

$$Aq = i\omega_0 q, \quad A\bar{q} = -i\omega_0 \bar{q}$$

(as in the fold case, its particular normalization is not important). Introduce also the *adjoint* eigenvector $p \in \mathbb{C}^n$ having the properties

$$A^T p = -i\omega_0 p, \quad A^T \bar{p} = i\omega_0 \bar{p},$$

and satisfying the normalization

$$\langle p, q \rangle = 1, \tag{5.52}$$

where $\langle p, q \rangle = \sum_{i=1}^{n} \bar{p}_i q_i$ is the standard scalar product in $\mathbb{C}^n$ (linear with respect to the second argument). The critical *real* eigenspace T^c corresponding to $\pm i\omega_0$ is now two-dimensional and is spanned by $\{\text{Re } q, \text{Im } q\}$. The real eigenspace T^{su} corresponding to all eigenvalues of A other than $\pm i\omega_0$ is $(n-2)$-dimensional. The following lemma is valid.

Lemma 5.3 *$y \in T^{su}$ if and only if $\langle p, y \rangle = 0$.* $\square$

Here $y \in \mathbb{R}^n$ is real, while $p \in \mathbb{C}^n$ is complex. Therefore, the condition in the lemma implies two real constraints on y (the real and imaginary parts of $\langle p, y \rangle$ must vanish). As in the previous case, this lemma allows us to decompose any $x \in \mathbb{R}^n$ as

$$x = zq + \bar{z}\bar{q} + y,$$

where $z \in \mathbb{C}^1$, and $zq + \bar{z}\bar{q} \in T^c$, $y \in T^{su}$. The complex variable z is a coordinate on T^c. We have

$$\left\{ \begin{array}{rcl} z & = & \langle p, x \rangle, \\ y & = & x - \langle p, x \rangle q - \langle \bar{p}, x \rangle \bar{q}. \end{array} \right. \tag{5.53}$$

(Notice that $\langle p, \bar{q} \rangle = 0$, see Lemma 3.3.) In the coordinates of (5.53), system (5.39) has the form

$$\left\{ \begin{array}{rcrcl} \dot{z} & = & i\omega_0 z & + & \langle p, F(zq + \bar{z}\bar{q} + y) \rangle, \\ \dot{y} & = & Ay & + & F(zq + \bar{z}\bar{q} + y) \\ & & & - & \langle p, F(zq + \bar{z}\bar{q} + y) \rangle q \\ & & & - & \langle \bar{p}, F(zq + \bar{z}\bar{q} + y) \rangle \bar{q}. \end{array} \right. \tag{5.54}$$

System (5.54) is $(n+2)$-dimensional, but one has to remember the two real constraints imposed on y. The system can now be written in a form similar to (5.26):

$$\left\{ \dot{z} = i\omega_0 z + \frac{1}{2} G_{20} z^2 + G_{11} z\bar{z} + \frac{1}{2} G_{02} \bar{z}^2 + \frac{1}{2} G_{21} z^2 \bar{z} \right.$$

constraints imposed on y. The system can now be written in a form similar to (5.26):

$$\begin{cases} \dot{z} & = & i\omega_0 z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z} \\ & & + \langle G_{10}, y\rangle z + \langle G_{01}, y\rangle \bar{z} + \cdots, \\ \dot{y} & = & Ay + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \cdots, \end{cases} \tag{5.55}$$

where $G_{20}, G_{11}, G_{02}, G_{21} \in \mathbb{C}^1$; $G_{01}, G_{10}, H_{ij} \in \mathbb{C}^n$; and the scalar product in $\mathbb{C}^n$ is used. Complex number and vectors involved in (5.55) can be computed by the following formulas:

$$\begin{aligned} G_{ij} &= \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j}\langle p, F(zq + \bar{z}\bar{q})\rangle\Big|_{z=0}, \quad i+j \geq 2, \\ \overline{G}_{10,i} &= \frac{\partial^2}{\partial y_i \partial z}\langle p, F(zq + \bar{z}\bar{q} + y)\rangle\Big|_{z=0,y=0}, \quad i = 1, 2, \ldots, n, \\ \overline{G}_{01,i} &= \frac{\partial^2}{\partial y_i \partial \bar{z}}\langle p, F(zq + \bar{z}\bar{q} + y)\rangle\Big|_{z=0,y=0}, \quad i = 1, 2, \ldots, n, \\ H_{ij} &= \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} F(zq + \bar{z}\bar{q})\Big|_{z=0} - G_{ij}q - \overline{G}_{ji}\bar{q}, \quad i+j = 2. \end{aligned}$$

The center manifold now has the representation

$$y = V(z, \bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2 + O(|z|^3),$$

where $\langle p, w_{ij} \rangle = 0$. The vectors $w_{ij} \in \mathbb{C}^n$ can be found from the linear equations

$$\begin{cases} (2i\omega_0 E - A)w_{20} &= H_{20}, \\ -Aw_{11} &= H_{11}, \\ (-2i\omega_0 E - A)w_{02} &= H_{02} \end{cases}$$

(cf. (5.31)). These equations have unique solutions since the matrices in their left-hand sides are *invertible* in the ordinary sense because $0, \pm 2i\omega_0$ are not eigenvalues of A. Thus, this case is even simpler than that of the fold, and the restricted equation can be written in the same way as (5.32):

$$\begin{aligned} \dot{z} &= i\omega_0 z + \tfrac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \tfrac{1}{2}G_{02}\bar{z}^2 \\ &+ \tfrac{1}{2}(G_{21} - 2\langle G_{10}, A^{-1}H_{11}\rangle + \langle G_{01}, (2i\omega_0 E - A)^{-1}H_{20}\rangle)z^2\bar{z} + \cdots, \end{aligned} \tag{5.56}$$

where the scalar product in $\mathbb{C}^n$ is used. A nice feature of the above algorithm is that it gives the restricted system (5.56) directly in the complex form suitable for the Lyapunov coefficient computations as described in Chapter 3.

As in the fold case, write $F(x)$ in terms of multilinear functions $B(x, y)$ and $C(x, y, z)$:

$$F(x) = \tfrac{1}{2}B(x, x) + \tfrac{1}{6}C(x, x, x) + O(\|x\|^4). \tag{5.57}$$

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$$\langle G_{10}, y\rangle = \langle p, B(q,y)\rangle,\ \ \langle G_{01}, y\rangle = \langle p, B(\bar{q},y)\rangle,$$

and write the restricted equation (5.56) in the form

$$\begin{aligned}\dot{z} \ &= \ i\omega_0 z + \tfrac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \tfrac{1}{2}G_{02}\bar{z}^2 \\ &\quad + \tfrac{1}{2}(G_{21} - 2\langle p, B(q, A^{-1}H_{11})\rangle + \langle p, B(\bar{q}, (2i\omega_0 E - A)^{-1}H_{20})\rangle)z^2\bar{z} \\ &\quad + \cdots,\end{aligned} \tag{5.58}$$

where

$$G_{20} = \langle p, B(q,q)\rangle,\ \ G_{11} = \langle p, B(q,\bar{q})\rangle,\ \ G_{02} = \langle p, B(\bar{q},\bar{q})\rangle, \tag{5.59}$$

$$G_{21} = \langle p, C(q,q,\bar{q})\rangle, \tag{5.60}$$

and

$$\begin{cases} H_{20} = B(q,q) - \langle p, B(q,q)\rangle q - \langle \bar{p}, B(q,q)\rangle \bar{q}, \\ H_{11} = B(q,\bar{q}) - \langle p, B(q,\bar{q})\rangle q - \langle \bar{p}, B(q,\bar{q})\rangle \bar{q}. \end{cases} \tag{5.61}$$

Substituting of (5.59)–(5.61) into (5.58), taking into account the identities

$$A^{-1}q = \frac{1}{i\omega_0}q,\ \ \ A^{-1}\bar{q} = -\frac{1}{i\omega_0}\bar{q},\ \ \ (2i\omega_0 E - A)^{-1}q = \frac{1}{i\omega_0}q,$$

$$(2i\omega_0 E - A)^{-1}\bar{q} = \frac{1}{3i\omega_0}\bar{q},$$

transforms (5.58) into the equation

$$\dot{z} = i\omega_0 z + \tfrac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \tfrac{1}{2}g_{02}\bar{z}^2 + \tfrac{1}{2}g_{21}z^2\bar{z} + \cdots,$$

where

$$g_{20} = \langle p, B(q,q)\rangle, \quad g_{11} = \langle p, B(q,\bar{q})\rangle,$$

and

$$\begin{aligned} g_{21} = {} & \langle p, C(q,q,\bar{q})\rangle \\ & - 2\langle p, B(q, A^{-1}B(q,\bar{q}))\rangle + \langle p, B(\bar{q}, (2i\omega_0 E - A)^{-1}B(q,q))\rangle \\ & + \frac{1}{i\omega_0}\langle p, B(q,q)\rangle\langle p, B(q,\bar{q})\rangle \\ & - \frac{2}{i\omega_0}\left|\langle p, B(q,\bar{q})\rangle\right|^2 - \frac{1}{3i\omega_0}\left|\langle p, B(\bar{q},\bar{q})\rangle\right|^2 . \end{aligned}$$

Notice that the terms in the last line are *purely imaginary* while the term in the third line contains the same scalar products as in the product $g_{20}g_{11}$. Thus, the application of formula (3.20) from Chapter 3,

$$l_1(0) = \frac{1}{2\omega_0^2}\,\mathrm{Re}(ig_{20}g_{11} + \omega_0 g_{21}),$$

Thus, the application of formula (3.20) from Chapter 3,

$$l_1(0) = \frac{1}{2\omega_0^2}\,\mathrm{Re}(ig_{20}g_{11} + \omega_0 g_{21}),$$

gives the following *invariant expression* for the first Lyapunov coefficient:

$$\begin{aligned} l_1(0) \quad &= \quad \frac{1}{2\omega_0}\mathrm{Re}\left[\langle p, C(q,q,\bar{q})\rangle - 2\langle p, B(q, A^{-1}B(q,\bar{q}))\rangle\right. \\ &\quad + \left.\langle p, B(\bar{q}, (2i\omega_0 E - A)^{-1}B(q,q))\rangle\right]. \end{aligned} \tag{5.62}$$

This formula seems to be the most convenient for analytical treatment of the Hopf bifurcation in n-dimensional systems with $n \geq 2$. It does not require a preliminary transformation of the system into its eigenbasis, and it expresses $l_1(0)$ using original linear, quadratic, and cubic terms, assuming that only the critical (ordinary and adjoint) eigenvectors of the Jacobian matrix are known. In Chapter 10 it will be shown how to implement this formula for the numerical evaluation of $l_1(0)$.

Example 5.4 (Hopf bifurcation in a feedback-control system) Consider the following nonlinear differential equation depending on positive parameters (α, β):

$$\frac{d^3y}{dt^3} + \alpha\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + y(1-y) = 0,$$

which describes a simple feedback control system of Lur'e type. By introducing $x_1 = y,\ x_2 = \dot{x}_1$, and $x_3 = \dot{x}_2$, we can rewrite the equation as the equivalent third-order system

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -\alpha x_3 - \beta x_2 - x_1 + x_1^2. \end{cases} \tag{5.63}$$

For all values of (α, β), system (5.63) has two equilibria $x^{(0)} = (0,0,0)$ and $x^{(1)} = (1,0,0)$. We will analyze the equilibrium at the origin. The Jacobian matrix of (5.63) evaluated at $x^{(0)}$ has the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -\beta & -\alpha \end{pmatrix}$$

with the characteristic equation

$$\lambda^3 + \alpha\lambda^2 + \beta\lambda + 1 = 0.$$

To find a relation between α and β corresponding to the Hopf bifurcation of $x^{(0)}$, substitute $\lambda = i\omega$ into the last equation. This shows that the characteristic polynomial has a pair of purely imaginary roots $\lambda_{1,2} = \pm i\omega,\ \omega > 0$, if

$$\alpha = \alpha_0(\beta) = \frac{1}{\beta},\ \beta > 0. \tag{5.64}$$

It is easy to check that the origin is stable if $\alpha > \alpha_0$ and unstable if $\alpha < \alpha_0$

if

$$\alpha = \alpha_0(\beta) = \frac{1}{\beta}, \quad \beta > 0. \tag{5.64}$$

It is easy to check that the origin is stable if $\alpha > \alpha_0$ and unstable if $\alpha < \alpha_0$. The transition is caused by a simple pair of complex-conjugate eigenvalues crossing the imaginary axis at $\lambda = \pm i\omega$, where

$$\omega^2 = \beta.$$

The velocity of the crossing is nonzero and the third eigenvalue λ_3 remains negative for nearby parameter values.[5] Thus, a Hopf bifurcation takes place. In order to analyze the bifurcation (i.e., to determine the direction of the limit cycle bifurcation), we have to compute the first Lyapunov coefficient $l_1(0)$ of the restricted system on the center manifold at the critical parameter values. If $l_1(0) < 0$, the bifurcation is supercritical and a unique stable limit cycle bifurcates from the origin for $\alpha < \alpha_0(\beta)$. As we shall see, this is indeed the case in system (5.63).

Therefore, fix α at its critical value α_0 given by (5.64) and leave β free to vary. Notice that the elements of the Jacobian matrix are rational functions of ω^2:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -\omega^2 & -1/\omega^2 \end{pmatrix}.$$

[5] At the critical parameter value (5.64), $\lambda_3 = -\frac{1}{\beta} < 0$.