1

Introduction to Dynamical Systems

This chapter introduces some basic terminology. First, we define a *dynamical system* and give several examples, including symbolic dynamics. Then we introduce the notions of *orbits, invariant sets*, and their *stability*. As we shall see while analyzing the *Smale horseshoe*, invariant sets can have

ical system and give several examples, including symbolic dynamics. Then we introduce the notions of *orbits, invariant sets*, and their *stability*. As we shall see while analyzing the *Smale horseshoe*, invariant sets can have very complex structures. This is closely related to the fact discovered in the 1960s that rather simple dynamical systems may behave "randomly," or "chaotically." Finally, we discuss how differential equations can define dynamical systems in both finite- and infinite-dimensional spaces.

1.1 Definition of a dynamical system

The notion of a dynamical system is the mathematical formalization of the general scientific concept of a *deterministic process*. The future and past states of many physical, chemical, biological, ecological, economical, and even social systems can be predicted to a certain extent by knowing their present state and the laws governing their evolution. Provided these laws do not change in time, the behavior of such a system could be considered as completely defined by its initial state. Thus, the notion of a dynamical system includes a set of its possible states (*state space*) and a law of the *evolution* of the state in *time*. Let us discuss these ingredients separately and then give a formal definition of a dynamical system.

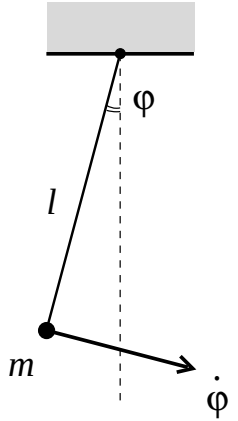


FIGURE 1.1. Classical pendulum.

1.1.1 State space

All possible states of a system are characterized by the points of some set X. This set is called the *state space* of the system. Actually, the specification of a point $x \in X$ must be sufficient not only to describe the current "position" of the system but also to determine its evolution. Different branches of science provide us with appropriate state spaces. Often, the state space is called a *phase space*, following a tradition from classical mechanics.

Example 1.1 (Pendulum) The state of an ideal pendulum is completely characterized by defining its angular *displacement* $\varphi \pmod{2\pi}$ from the vertical position and the corresponding angular *velocity* $\dot{\varphi}$ (see Figure 1.1). Notice that the angle φ alone is insufficient to determine the future

Example 1.1 (Pendulum) The state of an ideal pendulum is completely characterized by defining its angular *displacement* φ (mod 2π) from the vertical position and the corresponding angular *velocity* $\dot{\varphi}$ (see Figure 1.1). Notice that the angle φ alone is insufficient to determine the future state of the pendulum. Therefore, for this simple mechanical system, the state space is $X = \mathbb{S}^1 \times \mathbb{R}^1$, where $\mathbb{S}^1$ is the unit circle parametrized by the angle, and $\mathbb{R}^1$ is the real axis corresponding to the set of all possible velocities. The set X can be considered as a smooth two-dimensional *manifold* (cylinder) in $\mathbb{R}^3$. $\diamond$

Example 1.2 (General mechanical system) In classical mechanics, the state of an isolated system with s degrees of freedom is characterized by a $2s$-dimensional real vector:

$$(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s)^T,$$

where q_i are the *generalized coordinates*, while p_i are the corresponding *generalized momenta*. Therefore, in this case, $X = \mathbb{R}^{2s}$. If k coordinates are *cyclic*, $X = \mathbb{S}^k \times \mathbb{R}^{2s-k}$. In the case of the pendulum, $s = k = 1,\ q_1 = \varphi$, and we can take $p_1 = \dot{\varphi}$. $\diamond$

Example 1.3 (Quantum system) In quantum mechanics, the state of a system with *two observable states* is characterized by a vector

$$\psi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{C}^2,$$

where $a_i, i = 1, 2$, are complex numbers called *amplitudes*, satisfying the condition

$$|a_1|^2 + |a_2|^2 = 1.$$

The probability of finding the system in the ith state is equal to $p_i = |a_i|^2, i = 1, 2$. $\diamond$

Example 1.4 (Chemical reactor) The state of a well-mixed isothermic chemical reactor is defined by specifying the volume *concentrations* of the n reacting chemical substances

$$c = (c_1, c_2, \ldots, c_n)^T.$$

Clearly, the concentrations c_i must be nonnegative. Thus,

$$X = \{c : c = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n, c_i \geq 0\}.$$

If the concentrations change from point to point, the state of the reactor is defined by the reagent *distributions* $c_i(x), i = 1, 2, \ldots, n$. These functions are defined in a bounded spatial domain Ω, the reactor interior, and characterize the local concentrations of the substances near a point x. Therefore, the state space X in this case is a *function space* composed of vector-valued functions $c(x)$, satisfying certain smoothness and boundary conditions. $\diamond$

Example 1.5 (Ecological system) Similar to the previous example, the state of an ecological community within a certain domain Ω can be

functions $c(x)$, satisfying certain smoothness and boundary conditions. $\diamond$

Example 1.5 (Ecological system) Similar to the previous example, the state of an ecological community within a certain domain Ω can be described by a vector with nonnegative components

$$N = (N_1, N_2, \ldots, N_n)^T \in \mathbb{R}^n,$$

or by a vector function

$$N(x) = (N_1(x), N_2(x), \ldots, N_n(x))^T, \quad x \in \Omega,$$

depending on whether the spatial distribution is essential for an adequate description of the dynamics. Here N_i is the number (or density) of the ith species or other group (e.g., predators or prey). $\diamond$

Example 1.6 (Symbolic dynamics) To complete our list of state spaces, consider a set Ω_2 of all possible bi-infinite *sequences* of two symbols, say $\{1, 2\}$. A point $\omega \in X$ is the sequence

$$\omega = \{\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots\},$$

where $\omega_i \in \{1, 2\}$. Note that the zero position in a sequence must be pointed out; for example, there are *two* distinct periodic sequences that can be written as

$$\omega = \{\ldots, 1, 2, 1, 2, 1, 2, \ldots\},$$

one with $\omega_0 = 1$, and the other with $\omega_0 = 2$. The space Ω_2 will play an important role in the following.

Sometimes, it is useful to identify two sequences that differ only by a shift of the origin. Such sequences are called *equivalent*. The classes of equivalent sequences form a set denoted by $\widetilde{\Omega}_2$. The two periodic sequences mentioned above represent the same point in $\widetilde{\Omega}_2$. $\diamond$

In all the above examples, the state space has a certain natural structure, allowing for comparison between different states. More specifically, a *distance* ρ between two states is defined, making these sets *metric spaces*.

In the examples from mechanics and in the simplest examples from chemistry and ecology, the state space was a real vector space $\mathbb{R}^n$ of some finite dimension n, or a (sub-)manifold (hypersurface) in this space. The *Euclidean norm* can be used to measure the distance between two states parametrized by the points $x, y \in \mathbb{R}^n$, namely

$$\rho(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle} = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}, \qquad (1.1)$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{R}^n$,

$$\langle x, y\rangle = x^T y = \sum_{i=1}^{n} x_i y_i.$$

$$\langle x, y\rangle = x^T y = \sum_{i=1} x_i y_i.$$

If necessary, the distance between two (close) points on a manifold can be measured as the minimal length of a curve connecting these points within the manifold. Similarly, the distance between two states ψ, φ of the quantum system from Example 1.3 can be defined using the standard scalar product in $\mathbb{C}^n$,

$$\langle \psi, \varphi\rangle = \bar{\psi}^T \varphi = \sum_{i=1}^{n} \bar{\psi}_i \varphi_i,$$

with $n = 2$. Meanwhile, $\langle \psi, \psi\rangle = \langle \varphi, \varphi\rangle = 1$.

When the state space is a function space, there is a variety of possible distances, depending on the *smoothness* (differentiability) of the functions allowed. For example, we can introduce a distance between two continuous vector-valued real functions $u(x)$ and $v(x)$ defined in a bounded closed domain $\Omega \in \mathbb{R}^m$ by

$$\rho(u, v) = \|u - v\| = \max_{i=1,\ldots,n} \sup_{x\in\Omega} |u_i(x) - v_i(x)|.$$

Finally, in Example 1.6 the distance between two sequences $\omega, \theta \in \Omega_2$ can be measured by

$$\rho(\omega, \theta) = \sum_{k=-\infty}^{+\infty} \delta_{\omega_k \theta_k} 2^{-|k|}, \tag{1.2}$$

where

$$\delta_{\omega_k\theta_k} = \begin{cases} 0 & \text{if} \quad \omega_k = \theta_k, \\ 1 & \text{if} \quad \omega_k \neq \theta_k. \end{cases}$$

According to this formula, two sequences are considered to be close if they have a long block of coinciding elements centered at position zero (check!).

Using the previously defined distances, the introduced state spaces X are *complete metric spaces*. Loosely speaking, this means that any sequence of states, all of whose sufficiently future elements are separated by an arbitrarily small distance, is convergent (the space has no "holes").

According to the dimension of the underlying state space X, the dynamical system is called either *finite-* or *infinite-dimensional*. Usually, one distinguishes finite-dimensional systems defined in $X = \mathbb{R}^n$ from those defined on manifolds.

1.1.2 Time

The evolution of a dynamical system means a change in the state of the system with time $t \in T$, where T is a number set. We will consider two types of dynamical systems: those with continuous (real) time $T = \mathbb{R}^1$, and those with discrete (integer) time $T = \mathbb{Z}$. Systems of the first type are called *continuous-time* dynamical systems, while those of the second are termed *discrete-time* dynamical systems. Discrete-time systems appear

and those with discrete (integer) time $T = \mathbb{Z}$. Systems of the first type are called *continuous-time* dynamical systems, while those of the second are termed *discrete-time* dynamical systems. Discrete-time systems appear naturally in ecology and economics when the state of a system at a certain moment of time t completely determines its state after a year, say at $t+1$.

1.1.3 Evolution operator

The main component of a dynamical system is an evolution law that determines the state x_t of the system at time t, provided the *initial state* x_0 is known. The most general way to specify the evolution is to assume that for given $t \in T$ a map φ^t is defined in the state space X,

$$\varphi^t : X \to X,$$

which transforms an initial state $x_0 \in X$ into some state $x_t \in X$ at time t:

$$x_t = \varphi^t x_0.$$

The map φ^t is often called the *evolution operator* of the dynamical system. It might be known explicitly; however, in most cases, it is defined *indirectly* and can be computed only approximately. In the continuous-time case, the family $\{\varphi^t\}_{t \in T}$ of evolution operators is called a *flow*.

Note that $\varphi^t x$ may not be defined for all pairs $(x, t) \in X \times T$. Dynamical systems with evolution operator φ^t defined for both $t \geq 0$ and $t < 0$ are

called *invertible*. In such systems the initial state x_0 completely defines not only the *future* states of the system, but its *past* behavior as well. However, it is useful to consider also dynamical systems whose future behavior for $t > 0$ is completely determined by their initial state x_0 at $t = 0$, but the history for $t < 0$ can not be unambigously reconstructed. Such (*noninvertible*) dynamical systems are described by evolution operators defined only for $t \geq 0$ (i.e., for $t \in \mathbb{R}^1_+$ or $\mathbb{Z}_+$). In the continuous-time case, they are called *semiflows*.

It is also possible that $\varphi^t x_0$ is defined only *locally* in time, for example, for $0 \leq t < t_0$, where t_0 depends on $x_0 \in X$. An important example of such a behavior is a "blow-up," when a continuous-time system in $X = \mathbb{R}^n$ approaches infinity within a finite time, i.e.,

$$\|\varphi^t x_0\| \to +\infty,$$

for $t \to t_0$.

The evolution operators have two natural properties that reflect the deterministic character of the behavior of dynamical systems. First of all,

$$\varphi^0 = \mathrm{id}, \tag{DS.0}$$

where id is the identity map on X, $\mathrm{id}\, x = x$ for all $x \in X$. The property (DS.0) implies that the system does not change its state "spontaneously." The second property of the evolution operators reads

where id is the identity map on X, $\mathrm{id}\, x = x$ for all $x \in X$. The property (DS.0) implies that the system does not change its state "spontaneously." The second property of the evolution operators reads

$$\varphi^{t+s} = \varphi^t \circ \varphi^s. \tag{DS.1}$$

It means that

$$\varphi^{t+s}x = \varphi^t(\varphi^s x)$$

for all $x \in X$ and $t, s \in T$, *such that both sides of the last equation are defined*.[1] Essentially, the property (DS.1) states that the result of the evolution of the system in the course of $t + s$ units of time, starting at a point $x \in X$, is the same as if the system were first allowed to change from the state x over only s units of time and then evolved over the next t units of time from the resulting state $\varphi^s x$ (see Figure 1.2). This property means that the law governing the behavior of the system does not change in time: The system is "autonomous."

For invertible systems, the evolution operator φ^t satisfies the property (DS.1) for t and s both negative and nonnegative. In such systems, the operator φ^{-t} is the inverse to φ^t, $(\varphi^t)^{-1} = \varphi^{-t}$, since

$$\varphi^{-t} \circ \varphi^t = \mathrm{id}.$$

---

[1] Whenever possible, we will avoid explicit statements on the domain of definition of $\varphi^t x$.

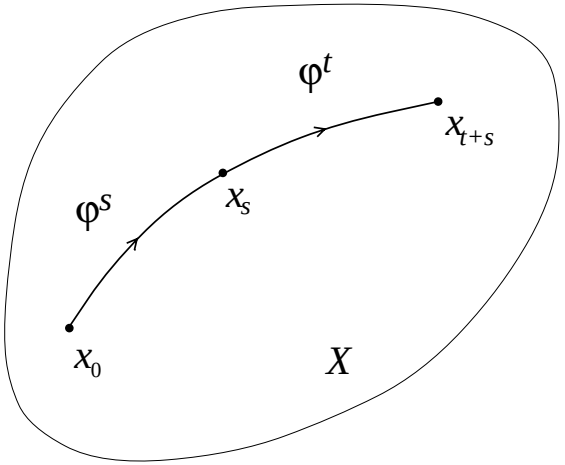


FIGURE 1.2. Evolution operator.

A discrete-time dynamical system with integer t is fully specified by defining only one map $f = \varphi^1$, called *"time-one map."* Indeed, using (DS.1), we obtain

$$\varphi^2 = \varphi^1 \circ \varphi^1 = f \circ f = f^2,$$

where f^2 is the *second iterate* of the map f. Similarly,

$$\varphi^k = f^k$$

where f^2 is the *second iterate* of the map f. Similarly,

$$\varphi^k = f^k$$

for all $k > 0$. If the discrete-time system is invertible, the above equation holds for $k \leq 0$, where $f^0 = \mathrm{id}$.

Finally, let us point out that, for many systems, $\varphi^t x$ is a continuous function of $x \in X$, and if $t \in \mathbb{R}^1$, it is also continuous in time. Here, the continuity is supposed to be defined with respect to the corresponding metric or norm in X. Furthermore, many systems defined on $\mathbb{R}^n$, or on smooth manifolds in $\mathbb{R}^n$, are such that $\varphi^t x$ is smooth as a function of (x, t). Such systems are called *smooth dynamical systems.*

1.1.4 Definition of a dynamical system

Now we are able to give a formal definition of a dynamical system.

Definition 1.1 *A* dynamical system *is a triple* $\{T, X, \varphi^t\}$*, where* T *is a time set,* X *is a state space, and* $\varphi^t : X \to X$ *is a family of evolution operators parametrized by* $t \in T$ *and satisfying the properties* (DS.0) *and* (DS.1)*.*

Let us illustrate the definition by two explicit examples.

Example 1.7 (A linear planar system) Consider the plane $X = \mathbb{R}^2$ and a family of linear nonsingular transformations on X given by the matrix

depending on $t \in \mathbb{R}^1$:

$$\varphi^t = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix},$$

where $\lambda, \mu \neq 0$ are real numbers. Obviously, it specifies a continuous-time dynamical system on X. The system is invertible and is defined for all (x, t). The map φ^t is continuous (and smooth) in x, as well as in t. $\diamond$

Example 1.8 (Symbolic dynamics) Take the space $X = \Omega_2$ of all bi-infinite sequences of two symbols $\{1, 2\}$ introduced in Example 1.6. Consider a map $\sigma : X \to X$, which transforms the sequence

$$\omega = \{\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots\} \in X$$

into the sequence $\theta = \sigma(\omega)$,

$$\theta = \{\ldots, \theta_{-2}, \theta_{-1}, \theta_0, \theta_1, \theta_2, \ldots\} \in X,$$

where

$$\theta_k = \omega_{k+1}, \quad k \in \mathbb{Z}.$$

The map σ merely shifts the sequence by one position to the left. It is called a *shift map*. The shift map defines a discrete-time dynamical system on X, $\varphi^k = \sigma^k$, that is invertible (find φ^{-1}). Notice that two sequences, θ and ω, are equivalent if and only if $\theta = \sigma^{k_0}(\omega)$ for some $k_0 \in \mathbb{Z}$. $\diamond$

Later on in the book, we will encounter many different examples of dy-

on X, $\varphi^{-1} = \sigma^{-1}$, that is invertible (find φ^{-1}). Notice that two sequences, θ and ω, are equivalent if and only if $\theta = \sigma^{k_0}(\omega)$ for some $k_0 \in \mathbb{Z}$. $\diamond$

Later on in the book, we will encounter many different examples of dynamical systems and will study them in detail.

1.2 Orbits and phase portraits

Throughout the book we use a geometrical point of view on dynamical systems. We shall always try to present their properties in geometrical images, since this facilitates their understanding. The basic geometrical objects associated with a dynamical system $\{T, X, \varphi^t\}$ are its *orbits* in the state space and the *phase portrait* composed of these orbits.

Definition 1.2 *An* orbit starting at x_0 *is an ordered subset of the state space* X*,*

$$Or(x_0) = \{x \in X : x = \varphi^t x_0, \text{ for all } t \in T \text{ such that } \varphi^t x_0 \text{ is defined}\}.$$

Orbits of a continuous-time system with a continuous evolution operator are *curves* in the state space X parametrized by the time t and oriented by its direction of increase (see Figure 1.3). Orbits of a discrete-time system are *sequences of points* in the state space X enumerated by increasing integers. Orbits are often also called *trajectories*. If $y_0 = \varphi^{t_0} x_0$ for some t_0, the sets $Or(x_0)$ and $Or(y_0)$ coincide. For example, two equivalent sequences

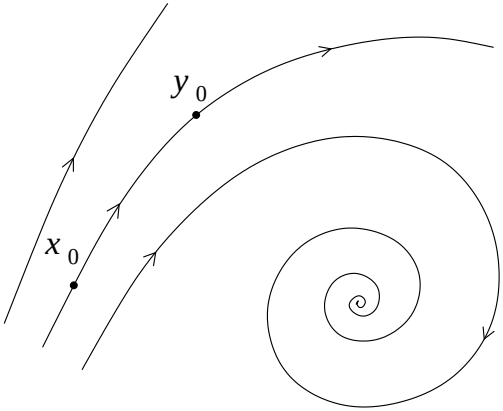


FIGURE 1.3. Orbits of a continuous-time system.

$\theta, \omega \in \Omega_2$ generate the same orbit of the symbolic dynamics $\{\mathbb{Z}, \Omega_2, \sigma^k\}$. Thus, all different orbits of the symbolic dynamics are represented by points in the set $\widetilde{\Omega}_2$ introduced in Example 1.6.

The simplest orbits are *equilibria*.

Definition 1.3 *A point $x^0 \in X$ is called an* equilibrium (fixed point) *if $\varphi^t x^0 = x^0$ for all $t \in T$.*

The evolution operator maps an equilibrium onto itself. Equivalently, a system placed at an equilibrium remains there forever. Thus, equilibria represent the simplest mode of behavior of the system. We will reserve the name "equilibrium" for continuous-time dynamical systems, while using

a system placed at an equilibrium remains there forever. Thus, equilibria represent the simplest mode of behavior of the system. We will reserve the name "equilibrium" for continuous-time dynamical systems, while using the term "fixed point" for corresponding objects of discrete-time systems. The system from Example 1.7 obviously has a single equilibrium at the origin, $x^0 = (0,0)^T$. If $\lambda, \mu < 0$, all orbits converge to x^0 as $t \to +\infty$ (this is the simplest mode of *asymptotic* behavior for large time). The symbolic dynamics from Example 1.7 have only two fixed points, represented by the sequences

$$\omega^1 = \{\ldots, 1, 1, 1, \ldots\}$$

and

$$\omega^2 = \{\ldots, 2, 2, 2, \ldots\}.$$

Clearly, the shift σ does not change these sequences: $\sigma(\omega^{1,2}) = \omega^{1,2}$.

Another relatively simple type of orbit is a *cycle*.

Definition 1.4 *A* cycle *is a periodic orbit, namely a nonequilibrium orbit L_0, such that each point $x_0 \in L_0$ satisfies $\varphi^{t+T_0} x_0 = \varphi^t x_0$ with some $T_0 > 0$, for all $t \in T$.*

The minimal T_0 with this property is called the *period* of the cycle L_0. If a system starts its evolution at a point x_0 on the cycle, it will return exactly to this point after every T_0 units of time. The system exhibits *periodic oscillations*. In the continuous-time case a cycle L_0 is a closed curve (see Figure 1.4(a)).

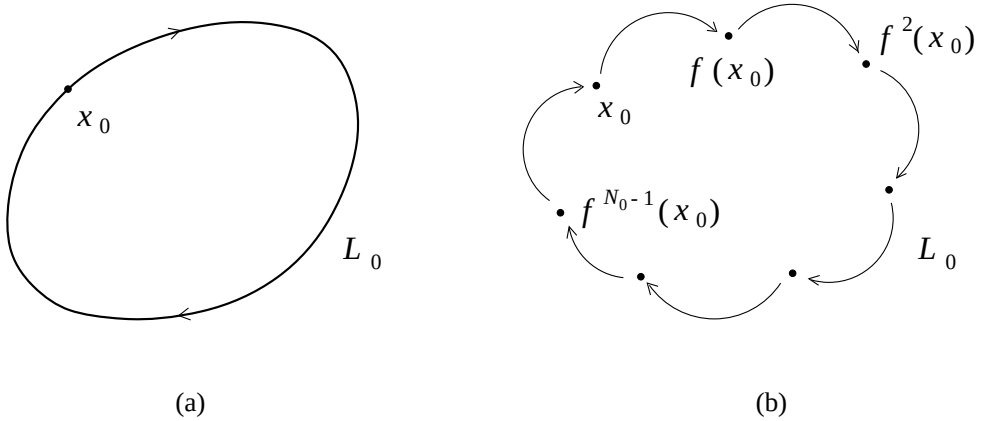


FIGURE 1.4. Periodic orbits in (a) a continuous-time and (b) a discrete-time system.

Definition 1.5 *A cycle of a continuous-time dynamical system, in a neighborhood of which there are no other cycles, is called a* limit cycle.

In the discrete-time case a cycle is a (finite) set of points

$$x_0, f(x_0), f^2(x_0), \ldots, f^{N_0}(x_0) = x_0,$$

where $f = \varphi^1$ and the period $T_0 = N_0$ is obviously an integer (Figure 1.4(b)). Notice that each point of this set is a *fixed point* of the N_0th iterate f^{N_0} of the map f. The system from Example 1.7 has no cycles. In contrast, the symbolic dynamics (Example 1.8) have an *infinite* number

1.4(b)). Notice that each point of this set is a *fixed point* of the N_0th iterate f^{N_0} of the map f. The system from Example 1.7 has no cycles. In contrast, the symbolic dynamics (Example 1.8) have an *infinite* number of cycles. Indeed, any *periodic sequence* composed of repeating blocks of length $N_0 > 1$ represents a cycle of period N_0, since we need to apply the shift σ exactly N_0 times to transform such a sequence into itself. Clearly, there is an infinite (though countable) number of such periodic sequences. Equivalent periodic sequences define the same periodic orbit.

We can roughly classify all possible orbits in dynamical systems into fixed points, cycles, and "all others."

Definition 1.6 *The* phase portrait *of a dynamical system is a partitioning of the state space into orbits.*

The phase portrait contains a lot of information on the behavior of a dynamical system. By looking at the phase portrait, we can determine the number and types of *asymptotic states* to which the system tends as $t \to +\infty$ (and as $t \to -\infty$ if the system is invertible). Of course, it is impossible to draw all orbits in a figure. In practice, only several key orbits are depicted in the diagrams to present phase portraits schematically (as we did in Figure 1.3). A phase portrait of a continuous-time dynamical system could be interpreted as an image of the flow of some fluid, where the orbits show the paths of "liquid particles" as they follow the current. This analogy explains the use of the term "flow" for the evolution operator in the continuous-time case.

1.3 Invariant sets

1.3.1 Definition and types

To further classify elements of a phase portrait – in particular, possible asymptotic states of the system – the following definition is useful.

Definition 1.7 *An* invariant set *of a dynamical system* $\{T, X, \varphi^t\}$ *is a subset* $S \subset X$ *such that* $x_0 \in S$ *implies* $\varphi^t x_0 \in S$ *for all* $t \in T$.

The definition means that $\varphi^t S \subseteq S$ for all $t \in T$. Clearly, an invariant set S consists of orbits of the dynamical system. Any individual orbit $Or(x_0)$ is obviously an invariant set. We always can *restrict* the evolution operator φ^t of the system to its invariant set S and consider a dynamical system $\{T, S, \psi^t\}$, where $\psi^t : S \to S$ is the map induced by φ^t in S. We will use the symbol φ^t for the restriction, instead of ψ^t.

If the state space X is endowed with a metric ρ, we could consider *closed invariant sets* in X. Equilibria (fixed points) and cycles are clearly the simplest examples of closed invariant sets. There are other types of closed invariant sets. The next more complex are *invariant manifolds*, that is, finite-dimensional hypersurfaces in some space $\mathbb{R}^K$. Figure 1.5 sketches an invariant two-dimensional *torus* $\mathbb{T}^2$ of a continuous-time dynamical system in $\mathbb{R}^3$ and a typical orbit on that manifold. One of the major discoveries in dynamical systems theory was the recognition that very simple, invertible,

invariant two-dimensional *torus* $\mathbb{T}^2$ of a continuous-time dynamical system in $\mathbb{R}^3$ and a typical orbit on that manifold. One of the major discoveries in dynamical systems theory was the recognition that very simple, invertible, differentiable dynamical systems can have extremely complex closed invariant sets containing an infinite number of periodic and nonperiodic orbits. Smale constructed the most famous example of such a system. It provides an invertible discrete-time dynamical system on the plane possessing an invariant set Λ, whose points are in one-to-one correspondence with all the bi-infinite sequences of two symbols. The invariant set Λ is not a manifold. Moreover, the restriction of the system to this invariant set behaves, in a certain sense, as the symbolic dynamics specified in Example 1.8. That is, how we can verify that it has an infinite number of cycles. Let us explore Smale's example in some detail.

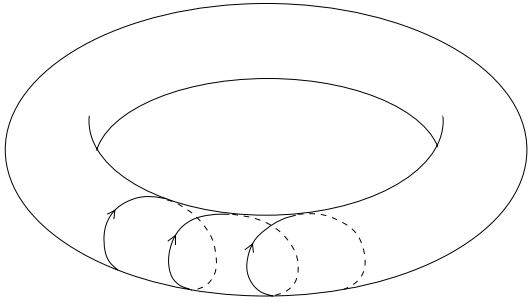


FIGURE 1.5. Invariant torus.

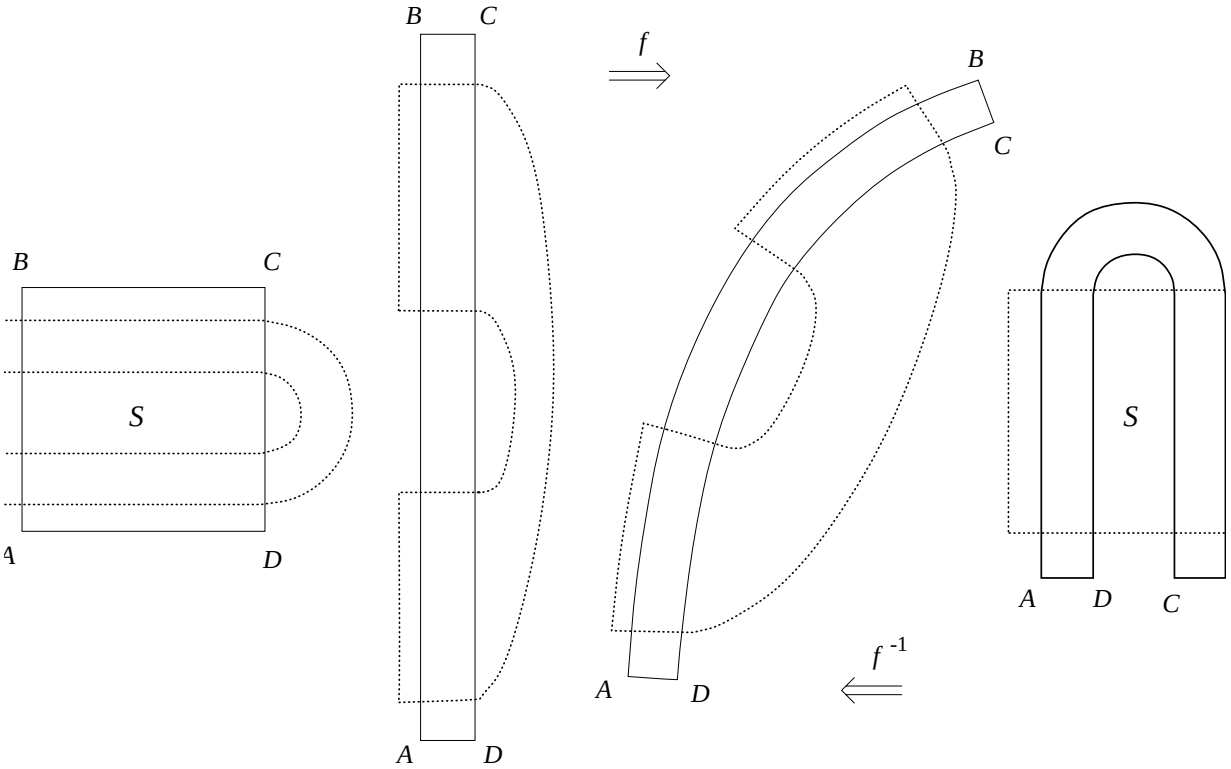


FIGURE 1.6. Construction of the horseshoe map.

1.3.2 Example 1.9 (Smale horseshoe)

FIGURE 1.6. Construction of the horseshoe map.

1.3.2 Example 1.9 (Smale horseshoe)

Consider the geometrical construction in Figure 1.6. Take a square S on the plane (Figure 1.6(a)). Contract it in the horizontal direction and expand it in the vertical direction (Figure 1.6(b)). Fold it in the middle (Figure 1.6(c)) and place it so that it intersects the original square S along two vertical strips (Figure 1.6(d)). This procedure defines a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. The image $f(S)$ of the square S under this transformation resembles a horseshoe. That is why it is called a *horseshoe map*. The exact shape of the image $f(S)$ is irrelevant; however, let us assume for simplicity that both the contraction and expansion are linear and that the vertical strips in the intersection are rectangles. The map f can be made invertible and smooth together with its inverse. The inverse map f^{-1} transforms the horseshoe $f(S)$ back into the square S through stages (d)–(a). This inverse transformation maps the dotted square S shown in Figure 1.6(d) into the dotted horizontal horseshoe in Figure 1.6(a), which we assume intersects the original square S along two horizontal rectangles.

Denote the vertical strips in the intersection $S \cap f(S)$ by V_1 and V_2,

$$S \cap f(S) = V_1 \cup V_1$$

(see Figure 1.7(a)). Now make the most important step: Perform the *second iteration* of the map f. Under this iteration, the vertical strips $V_{1,2}$ will be transformed into two "thin horseshoes" that intersect the square S along

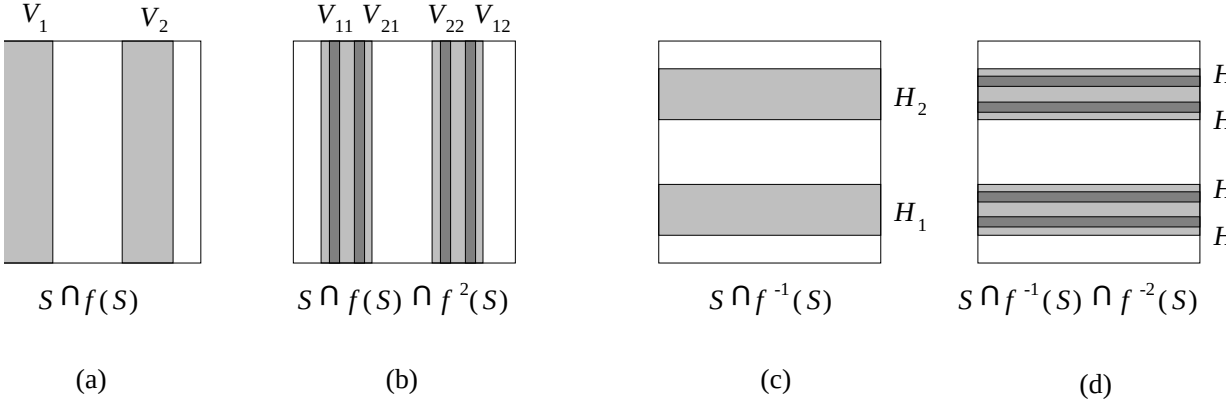


FIGURE 1.7. Vertical and horizontal strips.

four narrow vertical strips: V_{11}, V_{21}, V_{22}, and V_{12} (see Figure 1.7(b)). We write this as

$$S \cap f(S) \cap f^2(S) = V_{11} \cup V_{21} \cup V_{22} \cup V_{12}.$$

Similarly,

$$S \cap f^{-1}(S) = H_1 \cup H_2,$$

where $H_{1,2}$ are the horizontal strips shown in Figure 1.7(c), and

$$S \cap f^{-1}(S) \cap f^{-2}(S) = H_{11} \cup H_{12} \cup H_{22} \cup H_{21},$$

with four narrow horizontal strips H_{ij} (Figure 1.7(d)). Notice that $f(H_i) = V_i$, $i = 1, 2$, as well as $f^2(H_{ij}) = V_{ij}$, $i, j = 1, 2$ (Figure 1.8).

$S \cap f(S) = H_{11} \cup H_{12} \cup H_{22} \cup H_{21}$,

with four narrow horizontal strips H_{ij} (Figure 1.7(d)). Notice that $f(H_i) = V_i,\ i = 1, 2$, as well as $f^2(H_{ij}) = V_{ij},\ i, j = 1, 2$ (Figure 1.8).

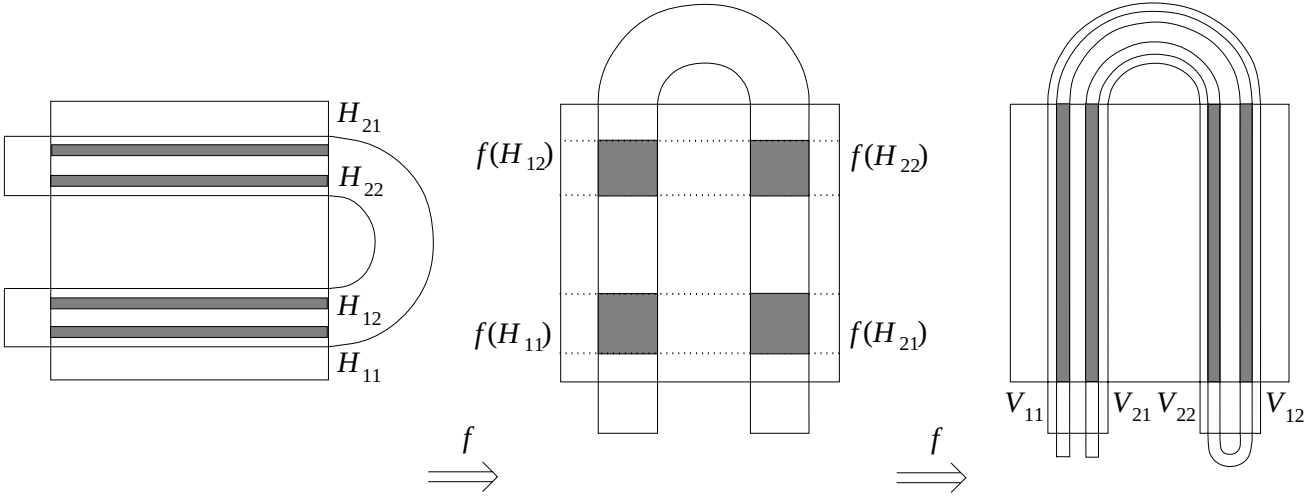


FIGURE 1.8. Transformation $f^2(H_{ij}) = V_{ij},\ i, j = 1, 2$.

Iterating the map f further, we obtain 2^k vertical strips in the intersection $S \cap f^k(S),\ k = 1, 2, \ldots$. Similarly, iteration of f^{-1} gives 2^k horizontal strips in the intersection $S \cap f^{-k}(S),\ k = 1, 2, \ldots$.

Most points leave the square S under iteration of f or f^{-1}. Forget about such points, and instead consider a set composed of all points in the plane

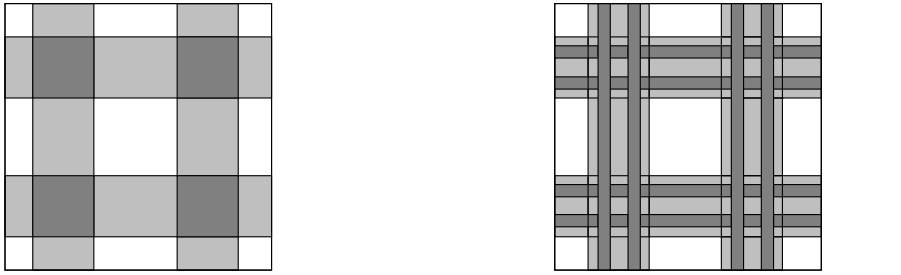


FIGURE 1.9. Location of the invariant set.

that remain in the square S under all iterations of f and f^{-1}:

$$\Lambda = \{x \in S : f^k(x) \in S, \text{ for all } k \in \mathbb{Z}\}.$$

Clearly, if the set Λ is nonempty, it is an *invariant set* of the discrete-time dynamical system defined by f. This set can be alternatively presented as an infinite intersection,

$$\Lambda = \cdots \cap f^{-k}(S) \cap \cdots \cap f^{-2}(S) \cap f^{-1}(S) \cap S \cap f(S) \cap f^2(S) \cap \cdots f^k(S) \cap \cdots$$

(any point $x \in \Lambda$ must belong to each of the involved sets). It is clear from this representation that the set Λ has a peculiar shape. Indeed, it should be located within

(any point $x \in \Lambda$ must belong to each of the involved sets). It is clear from this representation that the set Λ has a peculiar shape. Indeed, it should be located within

$$f^{-1}(S) \cap S \cap f(S),$$

which is formed by *four* small squares (see Figure 1.9(a)). Furthermore, it should be located inside

$$f^{-2}(S) \cap f^{-1}(S) \cap S \cap f(S) \cap f^2(S),$$

which is the union of *sixteen* smaller squares (Figure 1.9(b)), and so forth. In the limit, we obtain a *Cantor* (*fractal*) *set.*

Lemma 1.1 *There is a one-to-one correspondence $h : \Lambda \to \Omega_2$, between points of Λ and all bi-infinite sequences of two symbols.*

Proof:

For any point $x \in \Lambda$, define a sequence of two symbols $\{1, 2\}$

$$\omega = \{\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots\}$$

by the formula

$$\omega_k = \begin{cases} 1 & \text{if} \quad f^k(x) \in H_1, \\ 2 & \text{if} \quad f^k(x) \in H_2, \end{cases} \tag{1.3}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Here, $f^0 = \mathrm{id}$, the identity map. Clearly, this formula defines a map $h : \Lambda \to \Omega_2$, which assigns a sequence to each point of the invariant set.

To verify that this map is invertible, take a sequence $\omega \in \Omega_2$, fix $m > 0$, and consider a set $R_m(\omega)$ of all points $x \in S$, not necessarily belonging to Λ, such that

$$f^k(x) \in H_{\omega_k},$$

for $-m \le k \le m-1$. For example, if $m = 1$, the set R_1 is one of the four intersections $V_j \cap H_k$. In general, R_m belongs to the intersection of a vertical and a horizontal strip. These strips are getting thinner and thinner as $m \to +\infty$, approaching in the limit a vertical and a horizontal segment, respectively. Such segments obviously intersect at a single point x with $h(x) = \omega$. Thus, $h : \Lambda \to \Omega_2$ is a one-to-one map. It implies that Λ is nonempty. □

Remark:

The map $h : \Lambda \to \Omega_2$ is continuous together with its inverse (a *homeomorphism*) if we use the standard metric (1.1) in $S \subset \mathbb{R}^2$ and the metric given by (1.2) in Ω_2. ♢

Consider now a point $x \in \Lambda$ and its corresponding sequence $\omega = h(x)$, where h is the map previously constructed. Next, consider a point $y = f(x)$, that is, the image of x under the horseshoe map f. Since $y \in \Lambda$ by definition, there is a sequence that corresponds to y : $\theta = h(y)$. Is there a relation between these sequences ω and θ? As one can easily see from (1.3), such a relation exists and is very simple. Namely,

$$\theta_k = \omega_{k+1}, \quad k \in \mathbb{Z}.$$

between these sequences ω and θ? As one can easily see from (1.3), such a relation exists and is very simple. Namely,

$$\theta_k = \omega_{k+1}, \quad k \in \mathbb{Z},$$

since $f^k(f(x)) = f^{k+1}(x)$. In other words, the sequence θ can be obtained from the sequence ω by the *shift map* σ, defined in Example 1.8:

$$\theta = \sigma(\omega).$$

Therefore, the restriction of f to its invariant set $\Lambda \subset \mathbb{R}^2$ is *equivalent* to the shift map σ on the set of sequences Ω_2. Let us formulate this result as the following short lemma.

Lemma 1.2 *$h(f(x)) = \sigma(h(x))$, for all $x \in \Lambda$.*

We can write an even shorter one:

$$f|_\Lambda = h^{-1} \circ \sigma \circ h.$$

Combining Lemmas 1.1 and 1.2 with obvious properties of the shift dynamics on Ω_2, we get a theorem giving a rather complete description of the behavior of the horseshoe map.

Theorem 1.1 (Smale [1963]) *The horseshoe map f has a closed invariant set Λ that contains a countable set of periodic orbits of arbitrarily long period, and an uncountable set of nonperiodic orbits, among which there are orbits passing arbitrarily close to any point of Λ.* $\square$

The dynamics on Λ have certain features of "random motion." Indeed, for any sequence of two symbols we generate "randomly," thus prescribing the phase point to visit the horizontal strips H_1 and H_2 in a certain order, there is an orbit showing this feature among those composing Λ.

The next important feature of the horseshoe example is that we can slightly perturb the constructed map f without qualitative changes to its dynamics. Clearly, Smale's construction is based on a sufficiently strong contraction/expansion, combined with a folding. Thus, a (smooth) perturbation $\tilde{f}$ will have similar vertical and horizontal strips, which are no longer rectangles but curvilinear regions. However, provided the perturbation is sufficiently small (see the next chapter for precise definitions), these strips will shrink to *curves* that deviate only slightly from vertical and horizontal lines. Thus, the construction can be carried through verbatim, and the perturbed map $\tilde{f}$ will have an invariant set $\tilde{\Lambda}$ on which the dynamics are completely described by the shift map σ on the sequence space Ω_2. As we will discuss in Chapter 2, this is an example of *structurally stable* behavior.

Remark:

One can precisely specify the contraction/expansion properties required by the horseshoe map in terms of *expanding* and *contracting cones* of the Jacobian matrix f_x (see the literature cited in the bibliographical notes in Appendix 2 to this chapter). $\diamondsuit$

Appendix 2 to this chapter). ♢

1.3.3 Stability of invariant sets

To represent an observable asymptotic state of a dynamical system, an invariant set S_0 must be stable; in other words, it should "attract" nearby orbits. Suppose we have a dynamical system $\{T, X, \varphi^t\}$ with a complete metric state space X. Let S_0 be a closed invariant set.

Definition 1.8 *An invariant set S_0 is called* stable *if*

(i) *for any sufficiently small neighborhood $U \supset S_0$ there exists a neighborhood $V \supset S_0$ such that $\varphi^t x \in U$ for all $x \in V$ and all $t > 0$;*
(ii) *there exists a neighborhood $U_0 \supset S_0$ such that $\varphi^t x \to S_0$ for all $x \in U_0$, as $t \to +\infty$.*

If S_0 is an equilibrium or a cycle, this definition turns into the standard definition of stable equilibria or cycles. Property (i) of the definition is called *Lyapunov stability*. If a set S_0 is Lyapunov stable, nearby orbits do not leave its neighborhood. Property (ii) is sometimes called *asymptotic stability*. There are invariant sets that are Lyapunov stable but not asymptotically stable (see Figure 1.10(a)). In contrast, there are invariant sets that are attracting but not Lyapunov stable, since some orbits starting near S_0 eventually approach S_0, but only after an excursion outside a small but fixed neighborhood of this set (see Figure 1.10(b)).

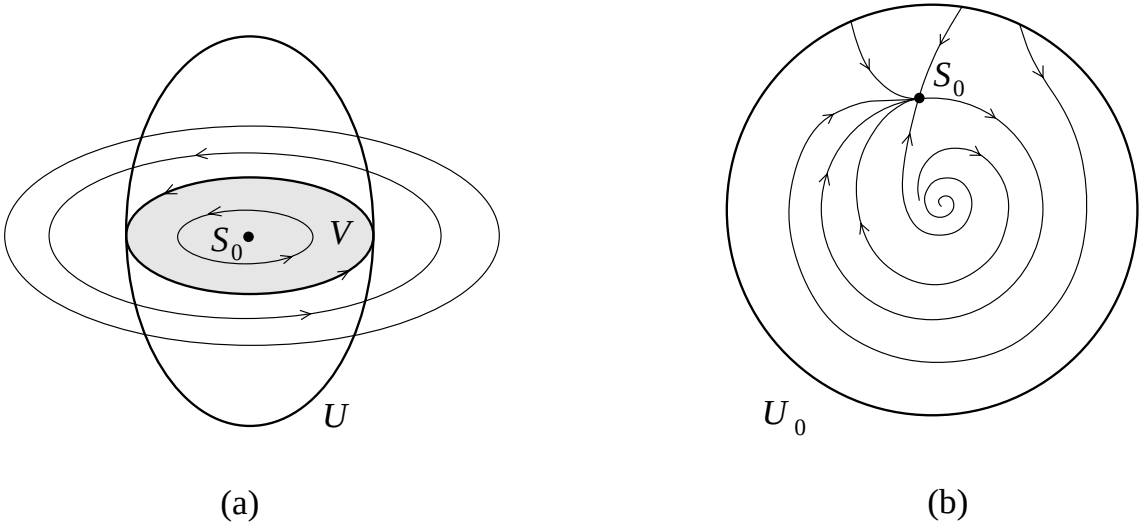


FIGURE 1.10. (a) Lyapunov versus (b) asymptotic stability.

If x^0 is a fixed point of a finite-dimensional, smooth, discrete-time dynamical system, then sufficient conditions for its stability can be formulated in terms of the Jacobian matrix evaluated at x^0.

Theorem 1.2 *Consider a discrete-time dynamical system*

$$x \mapsto f(x), \quad x \in \mathbb{R}^n,$$

where f is a smooth map. Suppose it has a fixed point x^0, namely $f(x^0) = x^0$, and denote by A the Jacobian matrix of $f(x)$ evaluated at $x^0, A = f_x(x^0)$. Then the fixed point is stable if all eigenvalues $\mu_1, \ldots, \mu_n$ of A

where f is a smooth map. Suppose it has a fixed point x^0, namely $f(x^0) = x^0$, and denote by A the Jacobian matrix of $f(x)$ evaluated at $x^0, A = f_x(x^0)$. Then the fixed point is stable if all eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of A satisfy $|\mu| < 1$. □

The eigenvalues of a fixed point are usually called *multipliers*. In the linear case the theorem is obvious from the Jordan normal form. Theorem 1.2, being applied to the N_0th iterate f^{N_0} of the map f at any point of the periodic orbit, also gives a sufficient condition for the stability of an N_0-cycle.

Another important case where we can establish the stability of a fixed point of a discrete-time dynamical system is provided by the following theorem.

Theorem 1.3 (Contraction Mapping Principle) *Let X be a complete metric space with distance defined by ρ. Assume that there is a map $f : X \to X$ that is continuous and that satisfies, for all $x, y \in X$,*

$$\rho(f(x), f(y)) \leq \lambda \rho(x, y),$$

with some $0 < \lambda < 1$. Then the discrete-time dynamical system $\{\mathbb{Z}_+, X, f^k\}$ has a stable fixed point $x^0 \in X$. Moreover, $f^k(x) \to x^0$ as $k \to +\infty$, starting from any point $x \in X$. □

The proof of this fundamental theorem can be found in any text on mathematical analysis or differential equations. Notice that there is no restric-

tion on the dimension of the space X: It can be, for example, an infinite-dimensional function space. Another important difference from Theorem 1.2 is that Theorem 1.3 *guarantees* the existence and uniqueness of the fixed point x^0, whereas this has to be *assumed* in Theorem 1.2. Actually, the map f from Theorem 1.2 is a contraction near x^0, provided an appropriate metric (norm) in $\mathbb{R}^n$ is introduced. The Contraction Mapping Principle is a powerful tool: Using this principle, we can prove the Implicit Function Theorem, the Inverse Function Theorem, as well as Theorem 1.4 ahead. We will apply the Contraction Mapping Principle in Chapter 4 to prove the existence, uniqueness, and stability of a closed invariant curve that appears under parameter variation from a fixed point of a generic planar map. Notice also that Theorem 1.3 gives *global* asymptotic stability: Any orbit of $\{\mathbb{Z}_+, X, f^k\}$ converges to x^0.

Finally, let us point out that the invariant set Λ of the horseshoe map is *not* stable. However, there are similar invariant fractal sets that are stable. Such objects are called *strange attractors*.

1.4 Differential equations and dynamical systems

The most common way to define a continuous-time dynamical system is by *differential equations*. Suppose that the state space of a system is $X = \mathbb{R}^n$ with coordinates $(x_1, x_2, \ldots, x_n)$. If the system is defined on a manifold, these can be considered as local coordinates on it. Very often the law of

The most common way to define a continuous-time dynamical system is by *differential equations*. Suppose that the state space of a system is $X = \mathbb{R}^n$ with coordinates $(x_1, x_2, \ldots, x_n)$. If the system is defined on a manifold, these can be considered as local coordinates on it. Very often the law of evolution of the system is given implicitly, in terms of the velocities $\dot{x}_i$ as functions of the coordinates $(x_1, x_2, \ldots, x_n)$:

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n,$$

or in the vector form

$$\dot{x} = f(x), \tag{1.4}$$

where the vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^n$ is supposed to be sufficiently differentiable (smooth). The function in the right-hand side of (1.4) is referred to as a *vector field*, since it assigns a vector $f(x)$ to each point x. Equation (1.4) represents a system of n *autonomous ordinary differential equations*, ODEs for short. Let us revisit some of the examples introduced earlier by presenting differential equations governing the evolution of the corresponding systems.

Example 1.1 (revisited) The dynamics of an ideal pendulum are described by Newton's second law,

$$\ddot{\varphi} = -k^2 \sin \varphi,$$

with

$$k^2 = \frac{g}{l},$$

where l is the pendulum length, and g is the gravity acceleration constant. If we introduce $\psi = \dot{\varphi}$, so that (φ, ψ) represents a point in the state space $X = \mathbb{S}^1 \times \mathbb{R}^1$, the above differential equation can be rewritten in the form of equation (1.4):

$$\left\{ \begin{array}{rcl} \dot{\varphi} & = & \psi, \\ \dot{\psi} & = & -k^2 \sin \varphi. \end{array} \right. \tag{1.5}$$

Here

$$x = \begin{pmatrix} \varphi \\ \psi \end{pmatrix},$$

while

$$f \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ -k^2 \sin \varphi \end{pmatrix}. \diamond$$

Example 1.2 (revisited) The behavior of an isolated energy-conserving mechanical system with s degrees of freedom is determined by $2s$ *Hamiltonian equations*:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{1.6}$$

for $i = 1, 2, \ldots, s$. Here the scalar function $H = H(q, p)$ is the *Hamilton function*. The equations of motion of the pendulum (1.5) are Hamiltonian equations with $(q, p) = (\varphi, \psi)$ and

$$H(\varphi, \psi) = \frac{\psi^2}{2} + k^2 \cos \varphi. \diamond$$

equations with $(q, p) = (\varphi, \psi)$ and

$$H(\varphi, \psi) = \frac{\psi^2}{2} + k^2 \cos \varphi. \diamond$$

Example 1.3 (revisited) The behavior of a quantum system with two states having different energies can be described between "observations" by the *Heisenberg equation*,

$$i\hbar \frac{d\psi}{dt} = H\psi,$$

where $i^2 = -1$,

$$\psi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad a_i \in \mathbb{C}^1.$$

The symmetric real matrix

$$H = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}, \quad E_0, A > 0,$$

is the Hamiltonian matrix of the system, and $\hbar$ is Plank's constant divided by 2π. The Heisenberg equation can be written as the following system of two linear *complex* equations for the amplitudes

$$\begin{cases} \dot{a}_1 &= \frac{1}{i\hbar}(E_0 a_1 - A a_2), \\ \dot{a}_2 &= \frac{1}{i\hbar}(-A a_1 + E_0 a_2). \diamond \end{cases} \tag{1.7}$$

Example 1.4 (revisited) As an example of a chemical system, let us consider the *Brusselator* [Lefever & Prigogine 1968]. This hypothetical system is composed of substances that react through the following irreversible stages:

$$\begin{aligned} A &\xrightarrow{k_1} X \\ B + X &\xrightarrow{k_2} Y + D \\ 2X + Y &\xrightarrow{k_3} 3X \\ X &\xrightarrow{k_4} E. \end{aligned}$$

Here capital letters denote reagents, while the constants k_i over the arrows indicate the corresponding reaction rates. The substances D and E do not re-enter the reaction, while A and B are assumed to remain constant. Thus, the *law of mass action* gives the following system of two nonlinear equations for the concentrations $[X]$ and $[Y]$:

$$\begin{aligned} \frac{d[X]}{dt} &= k_1[A] - k_2[B][X] - k_4[X] + k_3[X]^2[Y], \\ \frac{d[Y]}{dt} &= k_2[B][X] - k_3[X]^2[Y]. \end{aligned}$$

Linear scaling of the variables and time yields the *Brusselator equations*,

$$\left\{ \begin{aligned} \dot{x} &= a - (b+1)x + x^2 y, \\ \dot{y} &= bx - x^2 y. \end{aligned} \right. \quad \diamond \tag{1.8}$$

Linear scaling of the variables and time yields the *Brusselator equations*,

$$\left\{ \begin{array}{rcl} \dot{x} & = & a - (b+1)x + x^2 y, \\ \dot{y} & = & bx - x^2 y. \end{array} \right. \quad \diamond \tag{1.8}$$

Example 1.5 (revisited) One of the earliest models of ecosystems was the system of two nonlinear differential equations proposed by Volterra [1931]:

$$\left\{ \begin{array}{rcl} \dot{N}_1 & = & \alpha N_1 - \beta N_1 N_2, \\ \dot{N}_2 & = & -\gamma N_2 + \delta N_1 N_2. \end{array} \right. \tag{1.9}$$

Here N_1 and N_2 are the numbers of prey and predators, respectively, in an ecological community, α is the prey growth rate, γ is the predator mortality, while β and δ describe the predators' efficiency of consumption of the prey. $\diamond$

Under very general conditions, solutions of ODEs define smooth continuous-time dynamical systems. Few types of differential equations can be solved analytically (in terms of elementary functions). However, for smooth right-hand sides, the solutions are guaranteed to exist according to the following theorem. This theorem can be found in any textbook on ordinary differential equations. We formulate it without proof.

Theorem 1.4 (Existence, uniqueness, and smooth dependence) *Consider a system of ordinary differential equations*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth in an open region $U \subset \mathbb{R}^n$. Then there is a unique function $x = x(t, x_0)$, $x : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$, that is smooth in (t, x), and satisfies, for each $x_0 \in U$, the following conditions:

(i) $x(0, x_0) = x_0$;

(ii) *there is an interval $\mathcal{J} = (-\delta_1, \delta_2)$, where $\delta_{1,2} = \delta_{1,2}(x_0) > 0$, such that, for all $t \in \mathcal{J}$,*

$$y(t) = x(t, x_0) \in U,$$

and

$$\dot{y}(t) = f(y(t)). \ \square$$

The degree of smoothness of $x(t, x_0)$ with respect to x_0 in Theorem 1.4 is the same as that of f as a function of x. The function $x = x(t, x_0)$, considered as a function of time t, is called a *solution starting at* x_0. It defines, for each $x_0 \in U$, two objects: a *solution curve*

$$Cr(x_0) = \{(t, x) : x = x(t, x_0), t \in \mathcal{J}\} \subset \mathbb{R}^1 \times \mathbb{R}^n$$

and an *orbit*, which is the projection of $Cr(x_0)$ onto the state space,

$$Or(x_0) = \{x : x = x(t, x_0), t \in \mathcal{J}\} \subset \mathbb{R}^n$$

(see Figure 1.11). Both curves are parametrized by time t and oriented by the direction of time advance. A nonzero vector $f(x_0)$ is tangent to the orbit $Or(x_0)$ at x_0. There is a *unique* orbit passing through a point $x_0 \in U$

(see Figure 1.11). Both curves are parametrized by time t and oriented by the direction of time advance. A nonzero vector $f(x_0)$ is tangent to the orbit $Or(x_0)$ at x_0. There is a *unique* orbit passing through a point $x_0 \in U$.

Under the conditions of the theorem, the orbit either leaves U at $t = -\delta_1$ (and/or $t = \delta_2$), or stays in U forever; in the latter case, we can take $\mathcal{J} = (-\infty, +\infty)$.

Now we can define the evolution operator $\varphi^t : \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$\varphi^t x_0 = x(t, x_0),$$

which assigns to x_0 a point on the orbit through x_0 that is passed t time units later. Obviously, $\{\mathbb{R}^1, \mathbb{R}^n, \varphi^t\}$ is a continuous-time dynamical system (check!). This system is *invertible*. Each evolution operator φ^t is defined for $x \in U$ and $t \in \mathcal{J}$, where $\mathcal{J}$ depends on x_0 and is smooth in x. In practice, the evolution operator φ^t corresponding to a smooth system of ODEs can be found numerically on fixed time intervals to within desired accuracy. One of the standard ODE solvers can be used to accomplish this.

One of the major tasks of dynamical systems theory is to analyze the behavior of a dynamical system defined by ODEs. Of course, one might try to solve this problem by "brute force," merely computing many orbits numerically (by "simulations"). However, the most useful aspect of the theory is that we can predict some features of the phase portrait of a system defined by ODEs *without* actually solving the system. The simplest example of such information is the number and positions of equilibria. Indeed, the

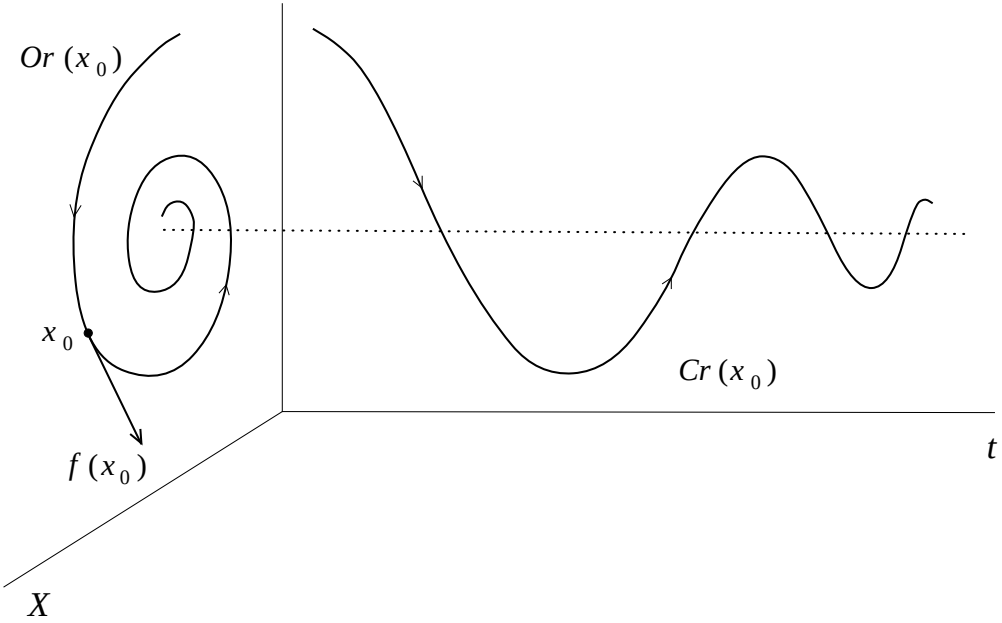


FIGURE 1.11. Solution curve and orbit.

equilibria of a system defined by (1.4) are zeros of the vector field given by its right-hand side:

$$f(x) = 0. \tag{1.10}$$

Clearly, if $f(x^0) = 0$, then $\varphi^t x_0 = x_0$ for all $t \in \mathbb{R}^1$. The stability of an equilibrium can also be detected without solving the system. For example, sufficient conditions for an equilibrium x^0 to be stable are provided by the following classical theorem.

equilibrium can also be detected without solving the system. For example, sufficient conditions for an equilibrium x^0 to be stable are provided by the following classical theorem.

Theorem 1.5 (Lyapunov [1892]) *Consider a dynamical system defined by*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

where f is smooth. Suppose that it has an equilibrium x^0 (i.e., $f(x^0) = 0$), and denote by A the Jacobian matrix of $f(x)$ evaluated at the equilibrium, $A = f_x(x^0)$. Then x^0 is stable if all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A satisfy $\operatorname{Re} \lambda < 0$. □

Recall that the eigenvalues are roots of the *characteristic equation*

$$\det(A - \lambda I) = 0,$$

where I is the $n \times n$ identity matrix.

The theorem can easily be proved for a linear system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n,$$

by its explicit solution in a basis where A has Jordan normal form, as well as for a general nonlinear system by constructing a *Lyapunov function* $L(x)$ near the equilibrium. More precisely, by a shift of coordinates, one can place the equilibrium at the origin, $x^0 = 0$, and find a certain quadratic form $L(x)$

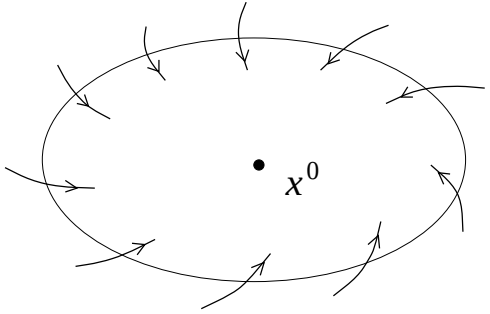


FIGURE 1.12. Lyapunov function.

whose level surfaces $L(x) = L_0$ surround the origin and are such that the vector field points strictly inside each level surface, sufficiently close to the equilibrium x^0 (see Figure 1.12). Actually, the Lyapunov function $L(x)$ is the same for both linear and nonlinear systems and is fully determined by the Jacobian matrix A. The details can be found in any standard text on differential equations (see the bibliographical notes in Appendix 2). Note also that the theorem can also be derived from Theorem 1.2 (see Exercise 7).

Unfortunately, in general it is impossible to tell by looking at the right-hand side of (1.4), whether this system has cycles (periodic solutions). Later on in the book we will formulate some efficient methods to prove the appearance of cycles under small perturbation of the system (e.g., by variation of parameters on which the system depends).

If the system has a smooth invariant manifold M, then its defining vector

the appearance of cycles under small perturbation of the system (e.g., by variation of parameters on which the system depends).

If the system has a smooth invariant manifold M, then its defining vector field $f(x)$ is *tangent* to M at any point $x \in M$, where $f(x) \neq 0$. For an $(n-1)$-dimensional smooth manifold $M \subset \mathbb{R}^n$, which is locally defined by $g(x) = 0$ for some scalar function $g : \mathbb{R}^n \to \mathbb{R}^1$, the invariance means

$$\langle \nabla g(x), f(x) \rangle = 0.$$

Here $\nabla g(x)$ denotes the *gradient*

$$\nabla g(x) = \left(\frac{\partial g(x)}{\partial x_1}, \frac{\partial g(x)}{\partial x_2}, \ldots, \frac{\partial g(x)}{\partial x_n} \right)^T,$$

which is orthogonal to M at x.

1.5 Poincaré maps

There are many cases where discrete-time dynamical systems (maps) naturally appear in the study of continuous-time dynamical systems defined by differential equations. The introduction of such maps allows us to apply the results concerning maps to differential equations. This is particularly efficient if the resulting map is defined in a lower-dimensional space than the original system. We will call maps arising from ODEs *Poincaré maps*.

1.5.1 Time-shift maps

The simplest way to extract a discrete-time dynamical system from a continuous-time system $\{\mathbb{R}^1, X, \varphi^t\}$ is to fix some $T_0 > 0$ and consider a system on X that is generated by iteration of the map $f = \varphi^{T_0}$. This map is called a *T_0-shift map* along orbits of $\{\mathbb{R}^1, X, \varphi^t\}$. Any invariant set of $\{\mathbb{R}^1, X, \varphi^t\}$ is an invariant set of the map f. For example, isolated fixed points of f are located at those positions where $\{\mathbb{R}^1, X, \varphi^t\}$ has isolated equilibria.

In this context, the *inverse* problem is more interesting: Is it possible to construct a system of ODEs whose T_0-shift map φ^{T_0} reproduces a given smooth and invertible map f? If we require the discrete-time system to have the same dimension as the continuous-time one, the answer is negative. The simplest counterexample is provided by the linear scalar map

$$x \mapsto f(x) = -\frac{1}{2}x, \quad x \in \mathbb{R}^1. \tag{1.11}$$

The map in (1.11) has a single fixed point $x^0 = 0$ that is stable. Clearly, there is no scalar ODE

$$\dot{x} = F(x), \quad x \in \mathbb{R}^1, \tag{1.12}$$

such that its evolution operator $\varphi^{T_0} = f$. Indeed, $x^0 = 0$ must be an equilibrium of (1.12), thus none of its orbits can "jump" over the origin like those of (1.11). We will return to this inverse problem in Chapter 9, where we explicitly construct ODE systems approximating certain maps.

equilibrium of (1.12), thus none of its orbits can "jump" over the origin like those of (1.11). We will return to this inverse problem in Chapter 9, where we explicitly construct ODE systems approximating certain maps.

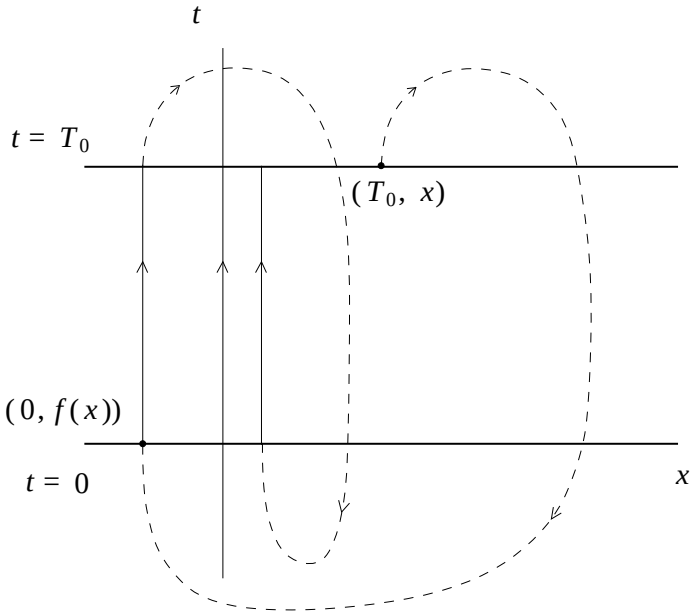


FIGURE 1.13. Suspension flow.

Remark:

If we allow for ODEs on *manifolds*, the inverse problem can always be solved. Specifically, consider a map $f : \mathbb{R}^n \to \mathbb{R}^n$ that is assumed to be

smooth, together with its inverse. Take a layer

$$\{(t,x) \in \mathbb{R}^1 \times \mathbb{R}^n : t \in [0, T_0]\}$$

(see Figure 1.13) and identify ("glue") a point (T_0, x) on the "top" face of X with the point $(0, f(x))$ on the "bottom" face. Thus, the constructed space X is an $(n+1)$-dimensional manifold with coordinates (t mod T_0, x). Specify now an autonomous system of ODEs on this manifold, called the *suspension*, by the equations

$$\left\{ \begin{array}{rcl} \dot{t} & = & 1, \\ \dot{x} & = & 0. \end{array} \right. \tag{1.13}$$

The orbits of (1.13) (viewed as subsets of $\mathbb{R}^1 \times \mathbb{R}^n$) are straight lines inside the layer interrupted by "jumps" from its "top" face to the "bottom" face. Obviously, the T_0-shift along orbits of (1.13) φ^{T_0} coincides on its invariant hyperplane $\{t = 0\}$ with the map f.

Let $k > 0$ satisfy the equation $e^{kT_0} = 2$. The suspension system corresponding to the map (1.11) has the same orbit structure as the system

$$\left\{ \begin{array}{rcl} \dot{t} & = & 1, \\ \dot{x} & = & -kx, \end{array} \right.$$

defined on an (infinitely wide) *Möbius strip* obtained by identifying the points (T_0, x) and $(0, -x)$ (see Figure 1.14). In both systems, $x = 0$ corresponds to a stable limit cycle of period T_0 with the multiplier $\mu = -\frac{1}{2}$.

defined on an (infinitely wide) *Möbius strip* obtained by identifying the points (T_0, x) and $(0, -x)$ (see Figure 1.14). In both systems, $x = 0$ corresponds to a stable limit cycle of period T_0 with the multiplier $\mu = -\frac{1}{2}$. $\diamondsuit$

1.5.2 Poincaré map and stability of cycles

Consider a continuous-time dynamical system defined by

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1.14}$$

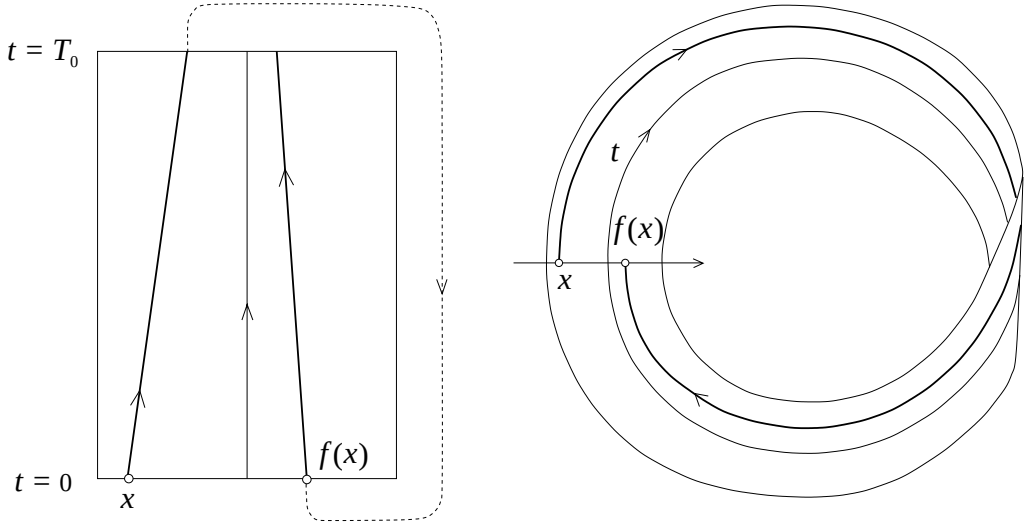


FIGURE 1.14. Stable limit cycle on the Möbius strip.

with smooth f. Assume, that (1.14) has a periodic orbit L_0. Take a point $x_0 \in L_0$ and introduce a *cross-section* Σ to the cycle at this point (see Figure 1.15). The cross-section Σ is a smooth hypersurface of dimension $n-1$, intersecting L_0 at a nonzero angle. Since the dimension of Σ is one less than the dimension of the state space, we say that the hypersurface Σ is of "codimension" one, codim $\Sigma = 1$. Suppose that Σ is defined near the point x_0 by the zero-level set of a smooth scalar function $g : \mathbb{R}^n \to \mathbb{R}^1,\ g(x_0) = 0$,

$$\Sigma = \{x \in \mathbb{R}^n : g(x) = 0\}.$$

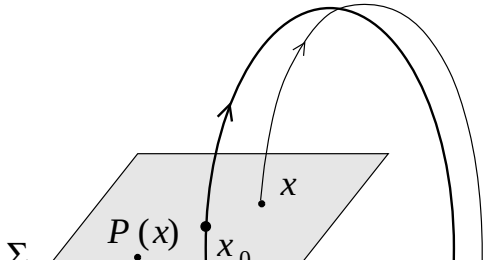
A nonzero intersection angle ("transversality") means that the gradient

$$\nabla g(x) = \left(\frac{\partial g(x)}{\partial x_1}, \frac{\partial g(x)}{\partial x_2}, \ldots, \frac{\partial g(x)}{\partial x_n}\right)^T$$

is not orthogonal to L_0 at x_0, that is,

$$\langle \nabla g(x_0), f(x_0) \rangle \neq 0,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. The simplest choice of Σ



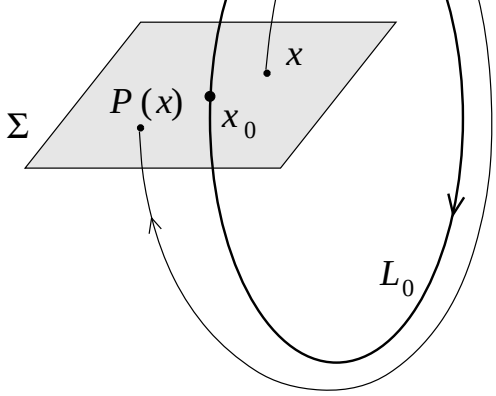


FIGURE 1.15. The Poincaré map associated with a cycle.

is a hyperplane orthogonal to the cycle L_0 at x_0. Such a cross-section is obviously given by the zero-level set of the linear function

$$g(x) = \langle f(x_0), x - x_0 \rangle.$$

Consider now orbits of (1.14) near the cycle L_0. The cycle itself is an orbit that starts at a point on Σ and returns to Σ at the same point ($x_0 \in \Sigma$). Since the solutions of (1.8) depend smoothly on their initial points (Theorem 1.4), an orbit starting at a point $x \in \Sigma$ sufficiently close to x_0 also returns to Σ at some point $\tilde{x} \in \Sigma$ near x_0. Moreover, nearby orbits will also intersect Σ transversally. Thus, a map $P : \Sigma \to \Sigma$,

$$x \mapsto \tilde{x} = P(x),$$

is constructed.

Definition 1.9 *The map P is called a* Poincaré map *associated with the cycle L_0.*

The Poincaré map P is locally defined, is as smooth as the right-hand side of (1.14), and is invertible near x_0. The invertibility follows from the invertibility of the dynamical system defined by (1.14). The inverse map $P^{-1} : \Sigma \to \Sigma$ can be constructed by extending the orbits crossing Σ backward in time until reaching their previous intersection with the cross-section. The intersection point x_0 is a *fixed point* of the Poincaré map: $P(x_0) = x_0$.

Let us introduce local coordinates $\xi = (\xi_1, \xi_2, \ldots, \xi_{n-1})$ on Σ such that $\xi = 0$ corresponds to x_0. Then the Poincaré map will be characterized by a locally defined map $P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, which transforms ξ corresponding to x into $\tilde{\xi}$ corresponding to $\tilde{x}$,

$$P(\xi) = \tilde{\xi}.$$

The origin $\xi = 0$ of $\mathbb{R}^{n-1}$ is a *fixed point* of the map P : $P(0) = 0$. The stability of the cycle L_0 is equivalent to the stability of the fixed point $\xi_0 = 0$ of the Poincaré map. Thus, the cycle is stable if all eigenvalues (multipliers) $\mu_1, \mu_2, \ldots, \mu_{n-1}$ of the $(n-1) \times (n-1)$ Jacobian matrix of P,

$$A = \left.\frac{dP}{d\xi}\right|_{\xi=0},$$

P,

$$A = \left.\frac{dP}{d\xi}\right|_{\xi=0},$$

are located inside the unit circle $|\mu| = 1$ (see Theorem 1.2).

One may ask whether the multipliers depend on the choice of the point x_0 on L_0, the cross-section Σ, or the coordinates ξ on it. If this were the case, determining stability using multipliers would be confusing or even impossible.

Lemma 1.3 *The multipliers $\mu_1, \mu_2, \ldots, \mu_{n-1}$ of the Jacobian matrix A of the Poincaré map P associated with a cycle L_0 are independent of the point x_0 on L_0, the cross-section Σ, and local coordinates on it.*

Proof:

Let Σ_1 and Σ_2 be two cross-sections to the same cycle L_0 at points x^1 and x^2, respectively (see Figure 1.16, where the planar case is presented for simplicity). We allow the points $x^{1,2}$ to coincide, and we let the cross-sections $\Sigma_{1,2}$ represent identical surfaces in $\mathbb{R}^n$ that differ only in parametrization. Denote by $P_1 : \Sigma_1 \to \Sigma_1$ and $P_2 : \Sigma_2 \to \Sigma_2$ corresponding Poincaré maps. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_{n-1})$ be coordinates on Σ_1, and let $\eta = (\eta_1, \eta_2, \ldots, \eta_{n-1})$ be coordinates on Σ_2, such that $\xi = 0$ corresponds to x^1 while $\eta = 0$ gives x^2. Finally, denote by A_1 and A_2 the associated Jacobian matrices of P_1 and P_2, respectively.

Due to the same arguments as those we used to construct the Poincaré map, there exists a locally defined, smooth, and invertible *correspondence*

map $Q : \Sigma_1 \to \Sigma_2$ along orbits of (1.14):

$$\eta = Q(\xi).$$

Obviously, we have

$$P_2 \circ Q = Q \circ P_1,$$

or, in coordinates,

$$P_2(Q(\xi)) = Q(P_1(\xi)),$$

for all sufficiently small $\|\xi\|$ (see Figure 1.15). Since Q is invertible, we obtain the following relation between P_1 and P_2:

$$P_1 = Q^{-1} \circ P_2 \circ Q.$$

Differentiating this equation with respect to ξ, and using the chain rule, we find

$$\frac{dP_1}{d\xi} = \frac{dQ^{-1}}{d\eta}\frac{dP_2}{d\eta}\frac{dQ}{d\xi}.$$

Evaluating the result at $\xi = 0$ gives the matrix equation

$$A_1 = B^{-1}A_2B,$$

where

$$B = \left.\frac{dQ}{d\xi}\right|_{\xi=0}$$

where

$$B = \left.\frac{dQ}{d\xi}\right|_{\xi=0}$$

is nonsingular (i.e., $\det B \neq 0$). Thus, the characteristic equations for A_1 and A_2 coincide, as do the multipliers. Indeed,

$$\det(A_1 - \mu I) = \det(B^{-1}) \det(A_2 - \mu I) \det(B) = \det(A_2 - \mu I),$$

since the determinant of the matrix product is equal to the product of the the determinants of the matrices involved, and $\det(B^{-1}) \det(B) = 1$. $\square$

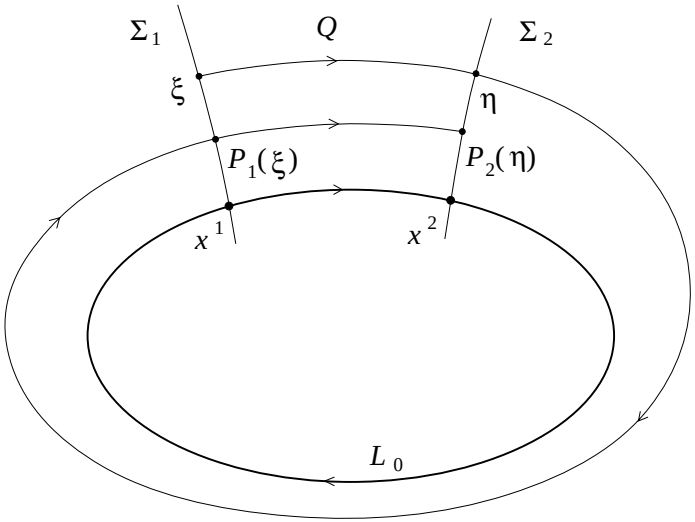


FIGURE 1.16. Two cross-sections to the cycle L_0.

According to Lemma 1.3, we can use *any* cross-section Σ to compute the multipliers of the cycle: The result will be the same.

The next problem to be addressed is the relationship between the multipliers of a cycle and the differential equations (1.14) defining the dynamical system that has this cycle. Let $x^0(t)$ denote a periodic solution of (1.14), $x^0(t+T_0) = x^0(t)$, corresponding to a cycle L_0. Represent a solution of (1.14) in the form

$$x(t) = x^0(t) + u(t),$$

where $u(t)$ is a deviation from the periodic solution. Then,

$$\dot{u}(t) = \dot{x}(t) - \dot{x}^0(t) = f(x^0(t) + u(t)) - f(x^0(t)) = A(t)u(t) + O(\|u(t)\|^2).$$

Truncating $O(\|u\|^2)$ terms results in the linear T_0-periodic system

$$\dot{u} = A(t)u, \quad u \in \mathbb{R}^n, \tag{1.15}$$

where $A(t) = f_x(x^0(t)),\ A(t+T_0) = A(t)$.

Definition 1.10 *System* (1.15) *is called the* variational equation *about the cycle* L_0.

The variational equation is the main (linear) part of the system governing the evolution of *perturbations* near the cycle. Naturally, the stability of the cycle depends on the properties of the variational equation.

Definition 1.11 *The time-dependent matrix* $M(t)$ *is called the* fundamen-

the evolution of *perturbations* near the cycle. Naturally, the stability of the cycle depends on the properties of the variational equation.

Definition 1.11 *The time-dependent matrix $M(t)$ is called the* fundamental matrix solution *of* (1.14) *if it satisfies*

$$\dot{M} = A(t)M,$$

with the initial condition $M(0) = I$, the identity $n \times n$ matrix.

Any solution $u(t)$ to (1.15) satisfies

$$u(T_0) = M(T_0)u(0)$$

(prove!). The matrix $M(T_0)$ is called a *monodromy matrix* of the cycle L_0. The following *Liouville formula* expresses the determinant of the monodromy matrix in terms of the matrix $A(t)$:

$$\det M(T_0) = \exp\left\{\int_0^{T_0} \operatorname{tr}\ A(t)\ dt\right\}. \tag{1.16}$$

Theorem 1.6 *The monodromy matrix $M(T_0)$ has eigenvalues*

$$1, \mu_1, \mu_2, \ldots, \mu_{n-1},$$

where μ_i are the multipliers of the Poincaré map associated with the cycle L_0.

Sketch of the proof:

Let φ^t be the evolution operator (flow) defined by system (1.14) near the cycle L_0. Consider the map

$$\varphi^{T_0} : \mathbb{R}^n \to \mathbb{R}^n.$$

Clearly, $\varphi^{T_0} x_0 = x_0$, where x_0 is an initial point on the cycle, which we assume to be located at the origin, $x_0 = 0$. The map is smooth, and its Jacobian matrix at x_0 coincides with the monodromy matrix:

$$\left.\frac{\partial \varphi^{T_0} x}{\partial x}\right|_{x=x_0} = M(T_0).$$

We claim that the matrix $M(T_0)$ has an eigenvalue $\mu_0 = 1$. Indeed, $v(t) = \dot{x}^0(t)$ is a solution to (1.15). Therefore, $q = v(0) = f(x_0)$ is transformed by $M(T_0)$ into itself:

$$M(T_0)q = q.$$

There are no generalized eigenvectors associated to q. Thus, the monodromy matrix $M(T_0)$ has a one-dimensional invariant subspace spanned by q and a complementary $(n-1)$-dimensional subspace $\Sigma : M(T_0)\Sigma = \Sigma$. Take the subspace Σ as a cross-section to the cycle at $x_0 = 0$. One can see that the restriction of the linear transformation defined by $M(T_0)$ to this invariant subspace Σ is the Jacobian matrix of the Poincaré map P defined by system (1.14) on Σ. Therefore, their eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n-1}$ coincide. □

this invariant subspace Σ is the Jacobian matrix of the Poincaré map P defined by system (1.14) on Σ. Therefore, their eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n-1}$ coincide. $\square$

According to (1.16), the product of all eigenvalues of $M(T_0)$ can be expressed as

$$\mu_1 \mu_2 \cdots \mu_{n-1} = \exp\left\{ \int_0^{T_0} (\text{div } f)(x^0(t))\, dt \right\}, \tag{1.17}$$

where, by definition, the *divergence* of a vector field $f(x)$ is given by

$$(\text{div } f)(x) = \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i}.$$

Thus, the product of all multipliers of any cycle is *positive*. Notice that in the planar case ($n = 2$) formula (1.17) allows us to compute the only multiplier μ_1, provided the periodic solution corresponding to the cycle is known explicitly. However, this is mainly a theoretical tool, since periodic solutions of nonlinear systems are rarely known analytically.

1.5.3 Poincaré map for periodically forced systems

In several applications the behavior of a system subjected to an external periodic forcing is described by *time-periodic* differential equations

$$\dot{x} = f(t, x), \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^n, \tag{1.18}$$

where $f(t+T_0,x)=f(t,x)$. System (1.18) defines an autonomous system on the cylindrical manifold $X=\mathbb{S}^1\times\mathbb{R}^n$, with coordinates $(t(\mathrm{mod}\ T_0),x)$, namely

$$\left\{\begin{array}{rcl}\dot{t} &=& 1,\\ \dot{x} &=& f(t,x).\end{array}\right. \tag{1.19}$$

In this space X, take the n-dimensional cross-section $\Sigma=\{(x,t)\in X : t=0\}$. We can use $x^T=(x_1,x_2,\ldots,x_n)$ as coordinates on Σ. Clearly, all orbits of (1.19) intersect Σ transversally. Assuming that the solution $x(t,x_0)$ of (1.19) exists on the interval $t\in[0,T_0]$, we can introduce the *Poincaré map*

$$x_0\mapsto P(x_0)=x(T_0,x_0).$$

In other words, we have to take an initial point x_0 and integrate system (1.18) over its period T_0 to obtain $P(x_0)$. By this construction, the discrete-time dynamical system $\{\mathbb{Z},\mathbb{R}^n,P^k\}$ is defined. Fixed points of P obviously correspond to T_0-periodic solutions of (1.18). An N_0-cycle of P represents an N_0T_0-periodic solution (*subharmonic*) of (1.18). The stability of these periodic solutions is clearly determined by that of the corresponding fixed points and cycles. More complicated solutions of (1.18) can also be studied via the Poincaré map. In Chapter 9 we will analyze in detail a model of a periodically (seasonally) forced predator-prey system exhibiting various subharmonic and chaotic solutions.

subharmonic and chaotic solutions.

1.6 Exercises

(1) (Symbolic dynamics and the Smale horseshoe revisited)
(a) Compute the number $N(k)$ of period-k cycles in the symbolic dynamics $\{\mathbb{Z}, \Omega_2, \sigma^k\}$.
(b) Explain how to find the coordinates of the two fixed points of the horseshoe map f in S. Prove that each point has one multiplier inside and one multiplier outside the unit circle $|\mu| = 1$.

(2) (Hamiltonian systems)
(a) Prove that the Hamilton function is constant along orbits of a Hamiltonian system: $\dot{H} = 0$.
(b) Prove that the equilibrium $(\varphi, \psi) = (0, 0)$ of a pendulum described by (1.5) is Lyapunov stable. (*Hint*: System (1.5) is Hamiltonian with closed level curves $H(\varphi, \psi) = \text{const}$ near $(0, 0)$.) Is this equilibrium asymptotically stable?

(3) (Quantum oscillations)
(a) Integrate the linear system (1.7), describing the simplest quantum system with two states, and show that the probability $p_i = |a_i|^2$ of finding the system in a given state oscillates periodically in time.

(b) How does $p_1 + p_2$ behave?

(4) (Brusselator revisited)

(a) Derive the Brusselator system (1.8) from the system written in terms of the concentrations $[X], [Y]$.

(b) Compute an equilibrium position (x_0, y_0) and find a sufficient condition on the parameters (a, b) for it to be stable.

(5) (Volterra system revisited)

(a) Show that (1.9) can be reduced by a linear scaling of variables and time to the following system with only one parameter γ:

$$\begin{cases} \dot{x} &= x - xy, \\ \dot{y} &= -\gamma y + xy. \end{cases}$$

(b) Find all equilibria of the scaled system.

(c) Verify that the orbits of the scaled system in the *positive quadrant* $\{(x, y) : x, y > 0\}$ coincide with those of the Hamiltonian system

$$\begin{aligned} \dot{x} &= \frac{1}{y} - 1, \\ \dot{y} &= -\frac{\gamma}{x} + 1. \end{aligned}$$

(*Hint*: Vector fields defining these two systems differ by the factor $\mu(x, y) =$

$$\dot{y} = -\frac{y}{x} + 1.$$

(*Hint*: Vector fields defining these two systems differ by the factor $\mu(x, y) = xy$, which is positive in the first quadrant.) Find the Hamilton function.

(d) Taking into account steps (a) to (c), prove that all nonequilibrium orbits of the Volterra system in the positive quadrant are closed, thus describing periodic oscillations of the numbers of prey and predators.

(6) (Explicit Poincaré map)

(a) Show that for $\alpha > 0$ the planar system in polar coordinates

$$\begin{cases} \dot{\rho} = \rho(\alpha - \rho^2), \\ \dot{\varphi} = 1, \end{cases}$$

has the explicit solution

$$\rho(t) = \left(\frac{1}{\alpha} + \left(\frac{1}{\rho_0^2} - \frac{1}{\alpha} \right) e^{-2\alpha t} \right)^{-1/2}, \ \varphi(t) = \varphi_0 + t.$$

(b) Draw the phase portrait of the system and prove that it has a unique limit cycle for each $\alpha > 0$.

(c) Compute the multiplier μ_1 of the limit cycle:

(i) by explicit construction of the Poincaré map $\rho \mapsto P(\rho)$ using the solution above and evaluating its derivative with respect to ρ at the fixed point $\rho_0 = \sqrt{\alpha}$ (*Hint:* See Wiggins [1990, pp. 66-67].);

(ii) using formula (1.17), expressing μ_1 in terms of the integral of the divergence over the cycle. (*Hint:* Use polar coordinates; the divergence is invariant.)

(7) (Lyapunov's theorem) Prove Theorem 1.5 using Theorem 1.2.

(a) Write the system near the equilibrium as

$$\dot{x} = Ax + F(x),$$

where $F(x) = O(\|x\|^2)$ is a smooth nonlinear function.

(b) Using the variation-of-constants formula for the evolution operator φ^t,

$$\varphi^t x = e^{At}x + \int_0^t e^{A(t-\tau)} F(\varphi^\tau x)\, d\tau,$$

show that the unit-time shift along the orbits has the expansion

$$\varphi^1 x = Bx + O(\|x\|^2),$$

where $B = e^A$.

(c) Conclude the proof, taking into account that $\mu_k = e^{\lambda_k}$, where μ_k and λ_k are the eigenvalues of the matrices B and A, respectively.

1.7 Appendix 1: Infinite-dimensional dynamical systems

λ_k are the eigenvalues of the matrices B and A, respectively.

1.7 Appendix 1: Infinite-dimensional dynamical systems defined by reaction-diffusion equations

As we have seen in Examples 1.4 and 1.5, the state of a spatially distributed system is characterized by a function from a *function space* X. The dimension of such spaces is *infinite*. A function $u \in X$ satisfies certain boundary and smoothness conditions, while its evolution is usually determined by a system of equations with *partial derivatives* (PDEs). In this appendix we briefly discuss how a particular type of such equations, namely *reaction-diffusion systems*, defines infinite-dimensional dynamical systems.

The state of a chemical reactor at time t can be specified by defining a vector function $c(x,t) = (c_1(x,t), c_2(x,t), \ldots, c_n(x,t))^T$, where the c_i are concentrations of reacting substances near the point x in the reactor domain $\Omega \subset \mathbb{R}^m$. Here $m = 1, 2, 3$, depending on the geometry of the reactor, and Ω is assumed to be closed and bounded by a smooth boundary $\partial\Omega$. The concentrations $c_i(x,t)$ satisfy certain problem-dependent *boundary conditions*. For example, if the concentrations of all the reagents are kept constant at the boundary, we have

$$c(x,t) = c_0, \quad x \in \partial\Omega.$$

Defining a deviation from the boundary value, $s(x,t) = c(x,t) - c_0$, we can reduce to the case of *zero Dirichlet boundary conditions*:

$$s(x,t) = 0, \quad x \in \partial\Omega.$$

If the reagents cannot penetrate the reactor boundary, *zero Neumann* (*zero flux*) *conditions* are applicable:

$$\frac{\partial c(x,t)}{\partial n} = 0, \quad x \in \partial\Omega,$$

where the left-hand side is the inward-pointing normal derivative at the boundary.

The evolution of a chemical system can be modeled by a system of *reaction-diffusion equations* written in the vector form for $u(x,t)$ ($u = s$ or c):

$$\frac{\partial u(x,t)}{\partial t} = D(\Delta u)(x,t) + f(u(x,t)), \tag{A.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth and D is a diagonal *diffusion matrix* with positive coefficients, and Δ is known as the *Laplacian*,

$$\Delta u = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2}.$$

The first term of the right-hand side of (A.1) describes diffusion of the

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

The first term of the right-hand side of (A.1) describes diffusion of the reagents, while the second term specifies their local interaction. The function $u(x,t)$ satisfies one of the boundary conditions listed above, for example, the Dirichlet conditions:

$$u(x,t) = 0, \quad x \in \partial\Omega. \tag{A.2}$$

Definition 1.12 *A function* $u = u(x,t),\ u : \Omega \times \mathbb{R}^1 \to \mathbb{R}^n$, *is called a* classical solution *to the problem* (A.1),(A.2) *if it is continuously differentiable, at least once with respect to* t *and twice with respect to* x, *and satisfies* (A.1),(A.2) *in the domain of its definition.*

For any twice continuously differentiable *initial function* $u_0(x)$,

$$u_0(x) = 0, \quad x \in \partial\Omega, \tag{A.3}$$

the problem (A.1),(A.2) has a unique classical solution $u(x,t)$, defined for $x \in \Omega$ and $t \in [0, \delta_0)$, where δ_0 depends on u_0, and such that $u(x,0) = u_0(x)$. Moreover, this classical solution is actually infinitely many times differentiable in (x,t) for $0 < t < \delta_0$. The same properties are valid if one replaces (A.2) by Neumann boundary conditions.

Introduce the space $X = C_0^2(\Omega, \mathbb{R}^n)$ of all twice continuously differentiable vector functions in Ω satisfying the Dirichlet condition (A.3) at the

boundary $\partial\Omega$. The preceeding results mean that the reaction-diffusion system (A.1),(A.2) defines a continuous-time dynamical system $\{\mathbb{R}^1_+, X, \varphi^t\}$, with the evolution operator

$$(\varphi^t u_0)(x) = u(x,t), \tag{A.4}$$

where $u(x,t)$ is the classical solution to (A.1),(A.2) satisfying $u(x,0) = u_0(x)$. It also defines a dynamical system on $X_1 = C_0^\infty(\Omega, \mathbb{R}^n)$ composed of all infinitely continuously differentiable vector functions in Ω satisfying the Dirichlet condition (A.3) at the boundary $\partial\Omega$.

The notions of equilibria and cycles are, therefore, applicable to the reaction-diffusion system (A.1). Clearly, equilibria of the system are described by time-independent vector functions satisfying

$$D(\Delta u)(x) + f(u(x)) = 0 \tag{A.5}$$

and the corresponding boundary conditions. A trivial, spatially homogeneous solutions to (A.5) satisfying (A.2), for example, is an equilibrium of the *local system*

$$\dot{u} = f(u), \quad u \in \mathbb{R}^n. \tag{A.6}$$

Nontrivial, spatially nonhomogeneous solutions to (A.5) are often called *dissipative structures*. Spatially homogeneous and nonhomogeneous equilibria can be stable or unstable. In the stable case, all (smooth) small perturbations $v(x)$ of an equilibrium solution decay in time. Cycles (i.e.,

dissipative structures. Spatially homogeneous and nonhomogeneous equilibria can be stable or unstable. In the stable case, all (smooth) small perturbations $v(x)$ of an equilibrium solution decay in time. Cycles (i.e., time-periodic solutions of (A.1) satisfying the appropriate boundary conditions) are also possible; they can be stable or unstable. Standing and rotating *waves* in reaction-diffusion systems in planar circular domains Ω are examples of such periodic solutions.

Up to now, the situation seems to be rather simple and is parallel to the finite-dimensional case. However, one runs into certain difficulties when trying to introduce a distance in $X = C_0^2(\Omega, \mathbb{R}^n)$. For example, this space is *incomplete* in the "integral norm"

$$\|u\|^2 = \int_\Omega \sum_{\substack{j=1,2,\ldots,n \\ |i|\le 2}} \left| \frac{\partial^{|i|} u_j(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_m^{i_m}} \right|^2 d\Omega, \tag{A.7}$$

where $|i| = i_1 + i_2 + \ldots + i_m$. In other words, a Cauchy sequence in this norm can approach a function that is not twice continuously differentiable (it may have no derivatives at all) and thus does not belong to X. Since this property is important in many respects, a method called *completion* has been developed that allows us to construct a complete space, given any normed one. Loosely speaking, we add the limits of all Cauchy sequences to X. More precisely, we call two Cauchy sequences *equivalent* if the distance between their corresponding elements tends to zero. Classes of equivalent

Cauchy sequences are considered as points of a new space H. The original norm can be extended to H, thus making it a complete normed space. Such spaces are called *Banach spaces*. The space X can then be interpreted as a subset of H. It is also useful if the obtained space is a *Hilbert space*, meaning that the norm in it is generated by a certain *scalar product*.

Therefore, we can try to use one of the completed spaces H as a new state space for our reaction-diffusion system. However, since H includes functions on which the diffusion part of (A.1) is undefined, extra work is required. One should also take care that the reaction part $f(u)$ of the system defines a smooth map on H. Without going into details, we merely state that it is possible to prove the existence of a dynamical system $\{\mathbb{R}^1_+, H, \psi^t\}$ such that $\psi^t u$ is defined and continuous in u for all $u \in H$ and $t \in [0, \delta(u))$, and, if $u_0 \in X \subset H$, then $\psi^t u_0 = \varphi^t u_0$, where $\varphi^t u_0$ is a classical solution to (A.1),(A.2).

The stability of equilibria and other solutions can be studied in the space H. If an equilibrium is stable in H, it will also be stable with respect to smooth perturbations. One can derive sufficient conditions for an equilibrium to be stable in H (or X) in terms of the linear part of the reaction-diffusion system (A.1). For example, let us formulate sufficient stability conditions (an analogue of Theorem 1.5) for a trivial (homogeneous) equilibrium of a reaction-diffusion system on the interval $\Omega = [0, \pi]$ with Dirichlet boundary conditions.

Theorem 1.7 *Consider a reaction-diffusion system*

with Dirichlet boundary conditions.

Theorem 1.7 *Consider a reaction-diffusion system*

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \tag{A.8}$$

where f is smooth, $x \in [0, \pi]$, with the boundary conditions

$$u(0) = u(\pi) = 0. \tag{A.9}$$

Assume that $u^0 = 0$ is a homogeneous equilibrium, $f(0) = 0$, and A is the Jacobian matrix of the corresponding equilibrium of the local system, $A = f_u(0)$. Suppose that the eigenvalues of the $n \times n$ matrix

$$M_k = A - k^2 D$$

have negative real parts for all $k = 0, 1, 2, \ldots$.

Then $u^0 = 0$ is a stable equilibrium of the dynamical system $\{\mathbb{R}^1_+, H, \psi^t\}$ generated by the system (A.8), (A.9) *in the completion H of the space $C_0^2([0, \pi], \mathbb{R}^n)$ in the norm* (A.7). □

A similar theorem can be proved for the system in $\Omega \subset \mathbb{R}^m, m = 2, 3$, with Dirichlet boundary conditions. The only modification is that k^2 should be replaced by κ_k, where $\{\kappa_k\}$ are all positive numbers for which

$$(\Delta v_k)(x) = -\kappa_k v_k(x),$$

with $v_k = v_k(x)$ satisfying Dirichlet boundary conditions. The modification to the Neumann boundary condition case is rather straightforward.

1.8 Appendix 2: Bibliographical notes

Originally, the term "dynamical system" meant only mechanical systems whose motion is described by differential equations derived in classical mechanics. Basic results on such dynamical systems were obtained by Lyapunov and Poincaré at the end of the nineteenth century. Their studies have been continued by Dulac [1923] and Birkhoff [1927], among others. The books by Nemytskii & Stepanov [1949] and Coddington & Levinson [1955] contain detailed treatments of the then-known properties of dynamical systems defined by differential equations. Later on it became clear that this notion is useful for the analysis of various evolutionary processes studied in different branches of science and described by ODEs, PDEs, or explicitly defined iterated maps. The modern period in dynamical system theory started from the work of Kolmogorov [1957], Smale [1963, 1966, 1967] and Anosov [1967]. Today, the literature on dynamical systems is huge. We do not attempt to survey it here, giving only a few remarks in the bibliographical notes to each chapter.

The horseshoe diffeomorphism proposed by Smale [1963, 1967] is treated in many books, for example, in Nitecki [1971], Guckenheimer & Holmes [1983], Wiggins [1990], Arrowsmith & Place [1990]. However, the best presentation of this and related topics is still due to Moser [1973].

General properties of ordinary differential equations and their relation to dynamical systems are presented in the cited book by Nemytskii and Stepanov, and notably in the texts by Pontryagin [1962], Arnold [1973],

General properties of ordinary differential equations and their relation to dynamical systems are presented in the cited book by Nemytskii and Stepanov, and notably in the texts by Pontryagin [1962], Arnold [1973], and Hirsch & Smale [1974]. The latter three books contain a comprehensive analysis of linear differential equations with constant and time-dependent coefficients. The book by Hartman [1964] treats the relation between Poincaré maps, multipliers, and stability of limit cycles.

The study of infinite-dimensional dynamical systems has been stimulated by hydro- and aerodynamics and by chemical and nuclear engineering. Linear infinite-dimensional dynamical systems, known as "continuous (analytical) semigroups" are studied in functional analysis (see, e.g., Hille & Phillips [1957], Balakrishnan [1976], or the more physically oriented texts by Richtmyer [1978, 1981]). The theory of nonlinear infinite-dimensional systems is a rapidly developing field. The reader is addressed to the relevant chapters of the books by Marsden & McCracken [1976], Carr [1981], and Henry [1981]. Infinite-dimensional dynamical systems also arise naturally in studying differential equations with delays (see Hale [1971], Hale & Verduyn Lunel [1993], and Diekmann, van Gils, Verduyn Lunel & Walther [1995]).

This page intentionally left blank

2

Topological Equivalence, Bifurcations, and Structural Stability of Dynamical Systems

In this chapter we introduce and discuss the following fundamental notions that will be used throughout the book: topological equivalence of dynamical systems and their classification, bifurcations and bifurcation diagrams, and topological normal forms for bifurcations. The last section is devoted to

that will be used throughout the book: topological equivalence of dynamical systems and their classification, bifurcations and bifurcation diagrams, and topological normal forms for bifurcations. The last section is devoted to the more abstract notion of structural stability. In this chapter we will be dealing only with dynamical systems in the state space $X = \mathbb{R}^n$.

2.1 Equivalence of dynamical systems

We would like to study general (qualitative) features of the behavior of dynamical systems, in particular, to classify possible types of their behavior and compare the behavior of different dynamical systems. The comparison of any objects is based on an *equivalence relation*,[1] allowing us to define classes of equivalent objects and to study transitions between these classes. Thus, we have to specify when we define two dynamical systems as being "qualitatively similar" or equivalent. Such a definition must meet some general intuitive criteria. For instance, it is natural to expect that two equivalent systems have the same number of equilibria and cycles of the same stability types. The "relative position" of these invariant sets and the

[1]Recall that a relation between two objects ($a \sim b$) is called *equivalence* if it is reflexive ($a \sim a$), symmetric ($a \sim b$ implies $b \sim a$), and transitive ($a \sim b$ and $b \sim c$ imply $a \sim c$).

shape of their regions of attraction should also be similar for equivalent systems. In other words, we consider two dynamical systems as equivalent if their phase portraits are "qualitatively similar," namely, if one portrait can be obtained from another by a continuous transformation (see Figure 2.1).

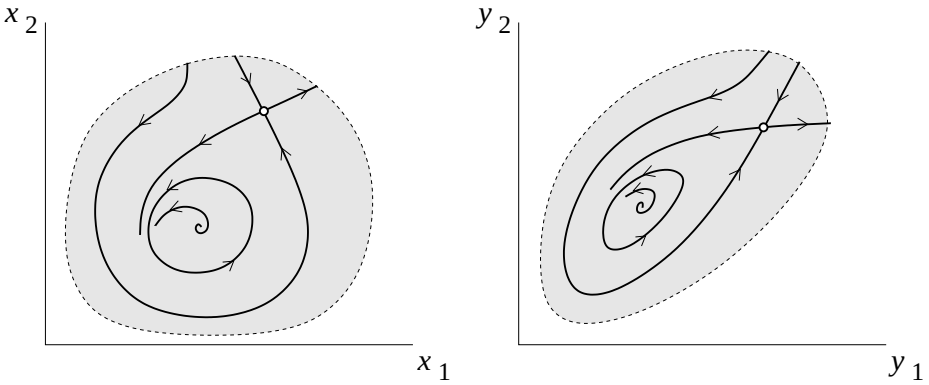


FIGURE 2.1. Topological equivalence.

Definition 2.1 *A dynamical system* $\{T, \mathbb{R}^n, \varphi^t\}$ *is called* topologically equivalent *to a dynamical system* $\{T, \mathbb{R}^n, \psi^t\}$ *if there is a homeomorphism* $h : \mathbb{R}^n \to \mathbb{R}^n$ *mapping orbits of the first system onto orbits of the second system, preserving the direction of time.*

A *homeomorphism* is an invertible map such that both the map and its inverse are continuous. The definition of the topological equivalence

system, preserving the direction of time.

A *homeomorphism* is an invertible map such that both the map and its inverse are continuous. The definition of the topological equivalence can be generalized to cover more general cases when the state space is a *complete metric* or, in particular, is a *Banach* space. The definition also remains meaningful when the state space is a smooth finite-dimensional *manifold* in $\mathbb{R}^n$, for example, a two-dimensional torus $\mathbb{T}^2$ or sphere $\mathbb{S}^2$. The phase portraits of topologically equivalent systems are often also called topologically equivalent.

The above definition applies to both continuous- and discrete-time systems. However, in the discrete-time case we can obtain an *explicit* relation between the corresponding maps of the equivalent systems. Indeed, let

$$x \mapsto f(x), \quad x \in \mathbb{R}^n, \tag{2.1}$$

and

$$y \mapsto g(y), \quad y \in \mathbb{R}^n, \tag{2.2}$$

be two topologically equivalent, discrete-time invertible dynamical systems ($f = \varphi^1, g = \psi^1$ are smooth invertible maps). Consider an orbit of system (2.1) starting at some point x:

$$\ldots,\ f^{-1}(x),\ x,\ f(x),\ f^2(x),\ \ldots$$

and an orbit of system (2.2) starting at a point y:

$$\ldots,\ g^{-1}(y),\ y,\ g(y),\ g^2(y),\ \ldots.$$

Topological equivalence implies that if x and y are related by the homeomorphism h, $y = h(x)$, then the first orbit is mapped onto the second one by this map h. Symbolically,

$$\begin{array}{ccc} x & \stackrel{f}{\longrightarrow} & f(x) \\ h\downarrow & & h\downarrow \\ y & \stackrel{g}{\longrightarrow} & g(y). \end{array}$$

Therefore, $g(y) = h(f(x))$ or $g(h(x)) = h(f(x))$ for all $x \in \mathbb{R}^n$, which can be written as

$$f(x) = h^{-1}(g(h(x)))$$

since h is invertible. We can write the last equation in a more compact form using the symbol of map superposition:

$$f = h^{-1} \circ g \circ h. \tag{2.3}$$

Definition 2.2 *Two maps f and g satisfying* (2.3) *for some homeomorphism h are called* conjugate.

Consequently, topologically equivalent, discrete-time systems are often called conjugate systems. If both h and h^{-1} are C^k maps, the maps f and g are called *C^k-conjugate*. For $k \geq 1$, *C^k-conjugate* maps (and the corresponding systems) are called *smoothly conjugate* or *diffeomorphic*. Two diffeomorphic maps (2.1) and (2.2) can be considered as the same map written

g are called C^k-*conjugate*. For $k \geq 1$, C^k-*conjugate* maps (and the corresponding systems) are called *smoothly conjugate* or *diffeomorphic*. Two diffeomorphic maps (2.1) and (2.2) can be considered as the same map written in two different coordinate systems with coordinates x and y, while $y = h(x)$ can be treated as a smooth *change of coordinates*. Consequently, diffeomorphic discrete-time dynamical systems are practically indistinguishable.

Now consider two continuous-time topologically equivalent systems:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{2.4}$$

and

$$\dot{y} = g(y), \quad y \in \mathbb{R}^n, \tag{2.5}$$

with smooth right-hand sides. Let φ^t and ψ^t denote the corresponding flows. In this case, there is no simple relation between f and g analogous to formula (2.3). Nevertheless, there are two particular cases of topological equivalence between (2.4) and (2.5) that can be expressed analytically, as we now explain.

Suppose that $y = h(x)$ is an invertible map $h : \mathbb{R}^n \to \mathbb{R}^n$, which is *smooth* together with its inverse (h is a *diffeomorphism*) and such that, for all $x \in \mathbb{R}^n$,

$$f(x) = M^{-1}(x)g(h(x)), \tag{2.6}$$

where

$$M(x) = \frac{dh(x)}{dx}$$

is the Jacobian matrix of $h(x)$ evaluated at the point x. Then, system (2.4) is topologically equivalent to system (2.5). Indeed, system (2.5) is obtained from system (2.4) by the smooth change of coordinates $y = h(x)$. Thus, h maps solutions of (2.4) into solutions of (2.5),

$$h(\varphi^t x) = \psi^t h(x),$$

and can play the role of the homeomorphism in Definition 2.1.

Definition 2.3 *Two systems* (2.4) *and* (2.5) *satisfying* (2.6) *for some diffeomorphism* h *are called* smoothly equivalent (*or* diffeomorphic).

Remark:

If the degree of smoothness of h is of interest, one writes: C^k-*equivalent* or C^k-*diffeomorphic* in Definition 2.3. $\diamondsuit$

Two diffeomorphic systems are practically identical and can be viewed as the same system written using different coordinates. For example, the eigenvalues of corresponding equilibria are the same. Let x_0 and $y_0 = h(x_0)$ be such equilibria and let $A(x_0)$ and $B(y_0)$ denote corresponding Jacobian matrices. Then, differentiation of (2.6) yields

$$A(x_0) = M^{-1}(x_0)B(y_0)M(x_0).$$

Therefore, the characteristic polynomials for the matrices $A(x_0)$ and $B(y_0)$ coincide. In addition, diffeomorphic limit cycles have the same multipliers

$$A(x_0) = M \quad (x_0)B(y_0)M(x_0).$$

Therefore, the characteristic polynomials for the matrices $A(x_0)$ and $B(y_0)$ coincide. In addition, diffeomorphic limit cycles have the same multipliers and period (see Exercise 4). This last property calls for more careful analysis of different *time parametrizations*.

Suppose that $\mu = \mu(x) > 0$ is a smooth *scalar positive* function and that the right-hand sides of (2.4) and (2.5) are related by

$$f(x) = \mu(x)g(x) \tag{2.7}$$

for all $x \in \mathbb{R}^n$. Then, obviously, systems (2.4) and (2.5) are topologically equivalent since their orbits are identical and it is the velocity of the motion that makes them different. (The ratio of the velocities at a point x is exactly $\mu(x)$.) Thus, the homeomorphism h in Definition 2.1 is the *identity* map $h(x) = x$. In other words, the systems are distinguished only by the time parametrization along the orbits.

Definition 2.4 *Two systems* (2.4) *and* (2.5) *satisfying* (2.7) *for a smooth positive function* μ *are called* orbitally equivalent.

Clearly, two orbitally equivalent systems can be nondiffeomorphic, having cycles that look like the same closed curve in the phase space but have different periods.

Very often we study system dynamics *locally*, e.g., not in the whole state space $\mathbb{R}^n$ but in some region $U \subset \mathbb{R}^n$. Such a region may be, for example, a

neighborhood of an equilibrium (fixed point) or a cycle. The above definitions of topological, smooth, and orbital equivalences can be easily "localized" by introducing appropriate regions. For example, in the topological classification of the phase portraits near equilibrium points, the following modification of Definition 2.1 is useful.

Definition 2.5 *A dynamical system* $\{T, \mathbb{R}^n, \varphi^t\}$ *is called* locally topologically equivalent *near an equilibrium* x_0 *to a dynamical system* $\{T, \mathbb{R}^n, \psi^t\}$ *near an equilibrium* y_0 *if there exists a homeomorphism* $h : \mathbb{R}^n \to \mathbb{R}^n$ *that is*

(i) *defined in a small neighborhood* $U \subset \mathbb{R}^n$ *of* x_0;

(ii) *satisfies* $y_0 = h(x_0)$;

(iii) *maps orbits of the first system in* U *onto orbits of the second system in* $V = f(U) \subset \mathbb{R}^n$, *preserving the direction of time.*

If U is an open neighborhood of x_0, then V is an open neighborhood of y_0. Let us also remark that equilibrium positions x_0 and y_0, as well as regions U and V, might coincide.

Let us compare the above introduced equivalences in the following example.

Example 2.1 (Node-focus equivalence) Consider two linear planar dynamical systems:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1, \\ \dot{x}_2 & = & -x_2, \end{array} \right. \tag{2.8}$$

dynamical systems:

$$\begin{cases} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -x_2, \end{cases} \tag{2.8}$$

and

$$\begin{cases} \dot{x}_1 &= -x_1 - x_2, \\ \dot{x}_2 &= x_1 - x_2. \end{cases} \tag{2.9}$$

In the polar coordinates (ρ, θ) these systems can be written as

$$\begin{cases} \dot{\rho} &= -\rho, \\ \dot{\theta} &= 0, \end{cases}$$

and

$$\begin{cases} \dot{\rho} &= -\rho, \\ \dot{\theta} &= 1, \end{cases}$$

respectively. Thus,

$$\begin{aligned} \rho(t) &= \rho_0 e^{-t}, \\ \theta(t) &= \theta_0, \end{aligned}$$

for the first system, while

$$\begin{aligned} \rho(t) &= \rho_0 e^{-t}, \\ \theta(t) &= \theta_0 + t, \end{aligned}$$

for the second. Clearly, the origin is a stable equilibrium in both systems, since $\rho(t) \to 0$ as $t \to \infty$. All other orbits of (2.8) are straight lines, while

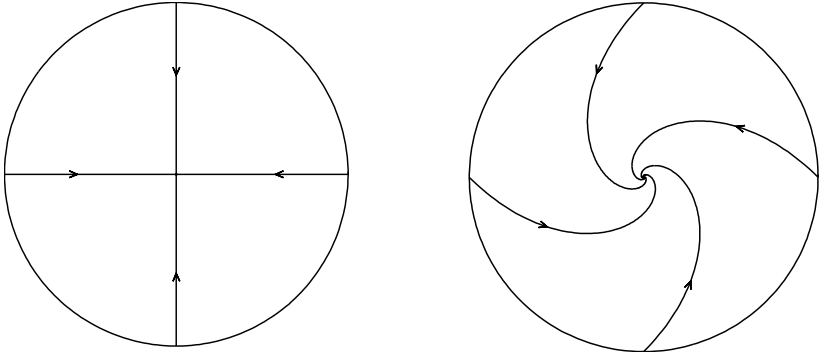


FIGURE 2.2. Node-focus equivalence.

those of (2.9) are spirals. The phase portraits of the systems are presented in Figure 2.2. The equilibrium of the first system is a *node* (Figure 2.2(a)), while in the second system it is a *focus* (Figure 2.2(b)). The difference in behavior of the systems can also be perceived by saying that perturbations decay near the origin monotonously in the first case and oscillatorily in the second case.

The systems are neither orbitally nor smoothly equivalent. The first fact is obvious, while the second follows from the observation that the eigenvalues of the equilibrium in the first system ($\lambda_1 = \lambda_2 = -1$) differ from those of the second ($\lambda_{1,2} = -1 \pm i$). Nevertheless, systems (2.8) and (2.9) are *topologically equivalent*, for example, in a closed unit disc

values of the equilibrium in the first system ($\lambda_1 = \lambda_2 = -1$) differ from those of the second ($\lambda_{1,2} = -1 \pm i$). Nevertheless, systems (2.8) and (2.9) are *topologically equivalent*, for example, in a closed unit disc

$$U = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} = \{(\rho, \theta) : \rho \leq 1\},$$

centered at the origin. Let us prove this explicitly by constructing a homeomorphism $h : U \to U$ as follows (see Figure 2.3). Take a point $x \neq 0$ in U with polar coordinates (ρ_0, θ_0) and consider the time τ required to move, along an orbit of system (2.8), from the point $(1, \theta_0)$ on the boundary to

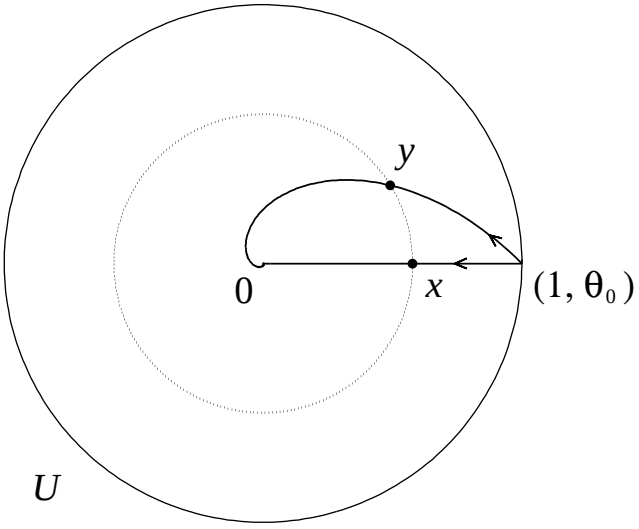


FIGURE 2.3. The construction of the homeomorphism.

the point x. This time depends only on ρ_0 and can easily be computed:

$$\tau(\rho_0) = -\ln \rho_0.$$

Now consider an orbit of system (2.9) starting at the boundary point $(1, \theta_0)$, and let $y = (\rho_1, \theta_1)$ be the point at which this orbit arrives after $\tau(\rho_0)$ units of time. Thus, a map $y = h(x)$ that transforms $x = (\rho_0, \theta_0) \neq 0$ into $y = (\rho_1, \theta_1)$ is obtained and is explicitly given by

$$h : \begin{cases} \rho_1 &= \rho_0, \\ \theta_1 &= \theta_0 - \ln \rho_0. \end{cases} \tag{2.10}$$

For $x = 0$, set $y = 0$, that is, $h(0) = 0$. Thus the constructed map transforms U into itself by rotating each circle $\rho_0 = \text{const}$ by a ρ_0-dependent angle. This angle equals zero at $\rho_0 = 1$ and increases as $\rho_0 \to 0$. The map is obviously continuous and invertible and maps orbits of (2.8) onto orbits of (2.9), preserving time direction. Thus, the two systems are topologically equivalent within U.

However, the homeomorphism h is not differentiable in U. More precisely, it is smooth away from the origin but not differentiable at $x = 0$. To see this, one should evaluate the Jacobian matrix $\frac{dy}{dx}$ in (x_1, x_2)-coordinates. For example, the difference quotient corresponding to the derivative

$$\left.\frac{\partial y_1}{\partial x_1}\right|_{x_1 = x_2 = 0}$$

is given for $x_1 > 0$ by

$$\left.\frac{\partial y_1}{\partial x_1}\right|_{x_1=x_2=0}$$

is given for $x_1 > 0$ by

$$\frac{x_1 \cos(\ln x_1) - 0}{x_1 - 0} = \cos(\ln x_1),$$

which has no limit as $x_1 \to 0$. $\diamond$

Therefore, considering continuous-time systems modulo topological equivalence, we preserve information on the number, stability, and topology of invariant sets, while losing information relating transient and time-dependent behavior. Such information may be important in some applications. In these cases, stronger equivalences (such as orbital or smooth) have to be applied.

A combination of smooth and orbital equivalences gives a useful equivalence relation, which will be used frequently in this book.

Definition 2.6 *Two systems* (2.4) *and* (2.5) *are called* smoothly orbitally equivalent *if* (2.5) *is smoothly equivalent to a system that is orbitally equivalent to* (2.4).

According to this definition, two systems are equivalent (in $\mathbb{R}^n$ or in some region $U \subset \mathbb{R}^n$) if we can transform one of them into the other by a smooth invertible change of coordinates and multiplication by a positive smooth function of the coordinates. Clearly, two smoothly orbitally equivalent systems are topologically equivalent, while the inverse is not true.

2.2 Topological classification of generic equilibria and fixed points

In this section we study the geometry of the phase portrait near *generic*, namely *hyperbolic*, equilibrium points in continuous- and discrete-time dynamical systems and present their topological classification.

2.2.1 Hyperbolic equilibria in continuous-time systems

Consider a continuous-time dynamical system defined by

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{2.11}$$

where f is smooth. Let $x_0 = 0$ be an equilibrium of the system (i.e., $f(x_0) = 0$) and let A denote the Jacobian matrix $\frac{df}{dx}$ evaluated at x_0. Let n_-, n_0, and n_+ be the numbers of eigenvalues of A (counting multiplicities) with negative, zero, and positive real part, respectively.

Definition 2.7 *An equilibrium is called* hyperbolic *if $n_0 = 0$, that is, if there are no eigenvalues on the imaginary axis. A hyperbolic equilibrium is called a* hyperbolic saddle *if $n_- n_+ \neq 0$.*

Since a generic matrix has no eigenvalues on the imaginary axis ($n_0 = 0$), hyperbolicity is a typical property and an equilibrium in a generic system (i.e., one not satisfying certain special conditions) is hyperbolic. We will not

Since a generic matrix has no eigenvalues on the imaginary axis ($n_0 = 0$), hyperbolicity is a typical property and an equilibrium in a generic system (i.e., one not satisfying certain special conditions) is hyperbolic. We will not try to formalize these intuitively obvious properties, though it is possible using measure theory and transversality arguments (see the bibliographical notes). Instead, let us study the geometry of the phase portrait near a hyperbolic equilibrium in detail. For an equilibrium (not necessarily a hyperbolic one), we introduce two invariant sets:

$$W^s(x_0) = \{x : \varphi^t x \to x_0, t \to +\infty\}, W^u(x_0) = \{x : \varphi^t x \to x_0, t \to -\infty\},$$

where φ^t is the flow associated with (2.11).

Definition 2.8 *$W^s(x_0)$ is called the* stable set *of x_0, while $W^u(x_0)$ is called the* unstable set *of x_0.*

Theorem 2.1 (Local Stable Manifold) *Let x_0 be a hyperbolic equilibrium (i.e., $n_0 = 0$, $n_- + n_+ = n$). Then the intersections of $W^s(x_0)$ and $W^u(x_0)$ with a sufficiently small neighborhood of x_0 contain smooth submanifolds $W^s_{\rm loc}(x_0)$ and $W^u_{\rm loc}(x_0)$ of dimension n_- and n_+, respectively.*

Moreover, $W^s_{\rm loc}(x_0)(W^u_{\rm loc}(x_0))$ is tangent at x_0 to $T^s(T^u)$, where $T^s(T^u)$ is the generalized eigenspace corresponding to the union of all eigenvalues of A with $\operatorname{Re}\lambda < 0$ ($\operatorname{Re}\lambda > 0$). $\square$

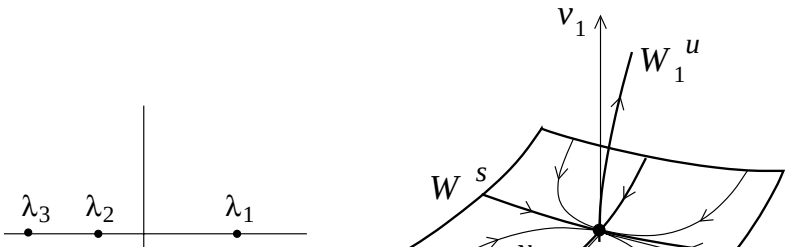
The proof of the theorem, which we are not going to present here, can be carried out along the following lines (Hadamard-Perron). For the unstable

manifold, take the linear manifold T^u passing through the equilibrium and apply the map φ^1 to this manifold, where φ^t is the flow corresponding to the system. The image of T^u under φ^1 is some (nonlinear) manifold of dimension n_+ tangent to T^u at x_0. Restrict attention to a sufficiently small neighborhood of the equilibrium where the linear part is "dominant" and repeat the procedure. It can be shown that the iterations converge to a smooth invariant submanifold defined in this neighborhood of x_0 and tangent to T^u at x_0. The limit is the local unstable manifold $W^u_{\rm loc}(x_0)$. The local stable manifold $W^s_{\rm loc}(x_0)$ can be constructed by applying φ^{-1} to T^s.

Remark:

Globally, the invariant sets W^s and W^u are *immersed* manifolds of dimensions n_- and n_+, respectively, and have the same smoothness properties as f. Having these properties in mind, we will call the sets W^s and W^u the *stable* and *unstable invariant manifolds of* x_0, respectively. $\diamondsuit$

Example 2.2 (Saddles and saddle-foci in $\mathbb{R}^3$) Figure 2.4 illustrates



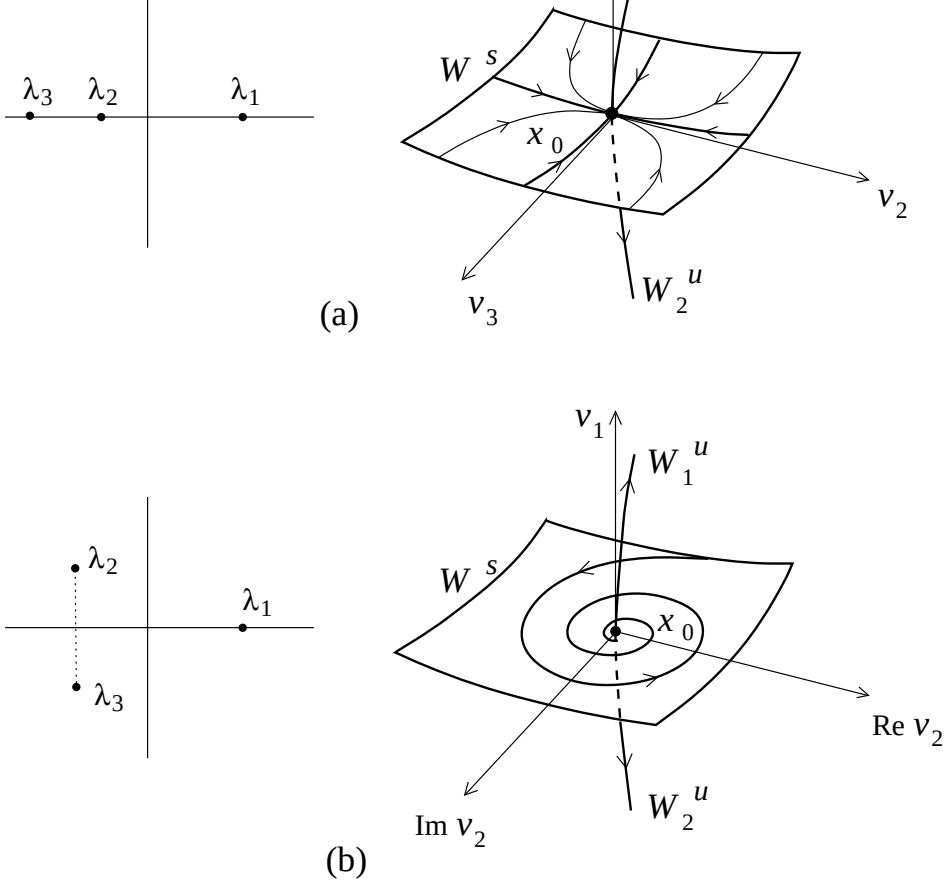


FIGURE 2.4. (a) Saddle and (b) saddle-focus: The vectors ν_k are the eigenvectors corresponding to the eigenvalues λ_k.

the theorem for the case where $n = 3$, $n_- = 2$, and $n_+ = 1$. In this

case, there are two invariant manifolds passing through the equilibrium, namely, the two-dimensional manifold $W^s(x_0)$ formed by all incoming orbits, and the one-dimensional manifold $W^u(x_0)$ formed by two outgoing orbits $W_1^u(x_0)$ and $W_2^u(x_0)$. All orbits not belonging to these manifolds pass near the equilibrium and eventually leave its neighborhood in both time directions.

In case (a) of real simple eigenvalues ($\lambda_3 < \lambda_2 < 0 < \lambda_1$), orbits on W^s form a node, while in case (b) of complex eigenvalues ($\mathrm{Re}\,\lambda_{2,3} < 0 < \lambda_1, \bar{\lambda}_3 = \lambda_2$), W^s carries a focus. Thus, in the first case, the equilibrium is called a *saddle*, while in the second one it is referred to as a *saddle-focus*. The equilibria in these two cases are topologically equivalent. Nevertheless, it is useful to distinguish them, as we shall see in our study of homoclinic orbit bifurcations (Chapter 6). $\diamond$

The following theorem gives the topological classification of hyperbolic equilibria.

Theorem 2.2 *The phase portraits of system* (2.11) *near two hyperbolic equilibria, x_0 and y_0, are locally topologically equivalent if and only if these equilibria have the same number n_- and n_+ of eigenvalues with* $\mathrm{Re}\,\lambda < 0$ *and with* $\mathrm{Re}\,\lambda > 0$*, respectively.* $\square$

Often, the equilibria x_0 and y_0 are then also called topologically equivalent. The proof of the theorem is based on two ideas. First, it is possible to show that near a hyperbolic equilibrium the system is locally topologi-

Often, the equilibria x_0 and y_0 are then also called topologically equivalent. The proof of the theorem is based on two ideas. First, it is possible to show that near a hyperbolic equilibrium the system is locally topologically equivalent to its *linearization*: $\dot{\xi} = A\xi$ (Grobman-Hartman Theorem). This result should be applied both near the equilibrium x_0 and near the equilibrium y_0. Second, the topological equivalence of two *linear* systems having the same numbers of eigenvalues with Re $\lambda < 0$ and Re $\lambda > 0$ and no eigenvalues on the imaginary axis has to be proved. Example 2.1 is a particular case of such a proof. Nevertheless, the general proof is based on the same idea. See the Appendix at the end of this chapter for references.

Example 2.3 (Generic equilibria of planar systems) Consider a two-dimensional system

$$\dot{x} = f(x), \quad x = (x_1, x_2)^T \in \mathbb{R}^2,$$

with smooth f. Suppose that $x = 0$ is an equilibrium, $f(0) = 0$, and let

$$A = \left.\frac{df(x)}{dx}\right|_{x=0}$$

be its Jacobian matrix. Matrix A has two eigenvalues λ_1, λ_2, which are the roots of the characteristic equation

$$\lambda^2 - \sigma\lambda + \Delta = 0,$$

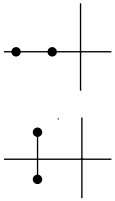
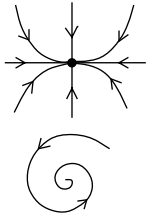
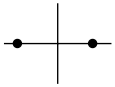
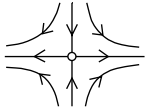
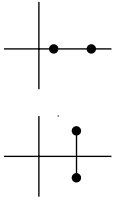
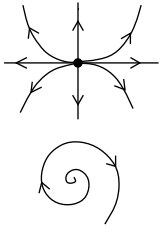
(n_+, n_-)	Eigenvalues	Phase portrait	Stability
(0, 2)		 node focus	stable
(1, 1)		 saddle	unstable
(2, 0)		 node focus	unstable

FIGURE 2.5. Topological classification of hyperbolic equilibria on the plane.

where $\sigma = \text{tr } A,\ \Delta = \det A$.

Figure 2.5 displays well-known classical results. There are three topological classes of hyperbolic equilibria on the plane: *stable nodes* (*foci*), *saddles, and unstable nodes* (*foci*). As we have discussed, nodes and foci (of corresponding stability) are topologically equivalent but can be identified

logical classes of hyperbolic equilibria on the plane: *stable nodes* (*foci*), *saddles, and unstable nodes* (*foci*). As we have discussed, nodes and foci (of corresponding stability) are topologically equivalent but can be identified looking at the eigenvalues.

Definition 2.9 *Nodes and foci are both called* antisaddles.

Stable points have two-dimensional stable manifolds and no unstable manifolds. For unstable equilibria the situation is reversed. Saddles have one-dimensional stable and unstable manifolds, sometimes called *separatrices.* $\diamond$

2.2.2 Hyperbolic fixed points in discrete-time systems

Now consider a discrete-time dynamical system

$$x \mapsto f(x), \quad x \in \mathbb{R}^n, \tag{2.12}$$

where the map f is smooth along with its inverse f^{-1} (diffeomorphism). Let $x_0 = 0$ be a fixed point of the system (i.e., $f(x_0) = x_0$) and let A denote the Jacobian matrix $\frac{df}{dx}$ evaluated at x_0. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of A are called *multipliers* of the fixed point. Notice that there are no zero multipliers, due to the invertibility of f. Let n_-, n_0, and n_+ be the numbers of multipliers of x_0 lying inside, on, and outside the unit circle $\{\mu \in \mathbb{C}^1 : |\mu| = 1\}$, respectively.

Definition 2.10 *A fixed point is called* hyperbolic *if* $n_0 = 0$, *that is, if there are no multipliers on the unit circle. A hyperbolic fixed point is called a* hyperbolic saddle *if* $n_- n_+ \neq 0$.

Notice that hyperbolicity is a typical property also in discrete time. As in the continuous-time case, we can introduce stable and unstable invariant sets for a fixed point x_0 (not necessarily a hyperbolic one):

$$\begin{aligned} W^s(x_0) &= \{x : f^k(x) \to x_0, k \to +\infty\}, \\ W^u(x_0) &= \{x : f^k(x) \to x_0, k \to -\infty\}, \end{aligned}$$

where k is integer "time" and $f^k(x)$ denotes the kth iterate of x under f. An analogue of Theorem 2.1 can be formulated.

Theorem 2.3 (Local Stable Manifold) *Let* x_0 *be a hyperbolic fixed point, namely,* $n_0 = 0$, $n_- + n_+ = n$. *Then the intersections of* $W^s(x_0)$ *and* $W^u(x_0)$ *with a sufficiently small neighborhood of* x_0 *contain smooth submanifolds* $W^s_{\rm loc}(x_0)$ *and* $W^u_{\rm loc}(x_0)$ *of dimension* n_- *and* n_+, *respectively.*

Moreover, $W^s_{\rm loc}(x_0)(W^u_{\rm loc}(x_0))$ *is tangent at* x_0 *to* $T^s(T^u)$, *where* $T^s(T^u)$ *is the generalized eigenspace corresponding to the union of all eigenvalues of* A *with* $|\mu| < 1 (|\mu| > 1)$. □

The proof of the theorem is completely analogous to that in the continuous-time case, if one substitutes φ^1 by f. Globally, the invariant sets W^s and W^u are again *immersed* manifolds of dimension n_- and n_+, re-

The proof of the theorem is completely analogous to that in the continuous-time case, if one substitutes φ^1 by f. Globally, the invariant sets W^s and W^u are again *immersed* manifolds of dimension n_- and n_+, respectively, and have the same smoothness properties as the map f. The manifolds cannot intersect themselves, but their global topology may be very complex, as we shall see later.

The topological classification of hyperbolic fixed points follows from a theorem that is similar to Theorem 2.2 for equilibria in the continuous-time systems.

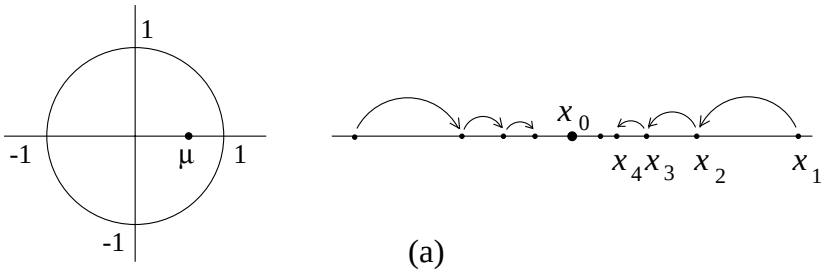
Theorem 2.4 *The phase portraits of* (2.12) *near two hyperbolic fixed points,* x_0 *and* y_0*, are locally topologically equivalent if and only if these fixed points have the same number* n_- *and* n_+ *of multipliers with* $|\mu| < 1$ *and* $|\mu| > 1$*, respectively, and the signs of the products of all the multipliers with* $|\mu| < 1$ *and with* $|\mu| > 1$ *are the same for both fixed points.* □

As in the continuous-time case, the proof is based upon the fact that near a hyperbolic fixed point the system is locally topologically equivalent to its *linearization*: $x \mapsto Ax$ (discrete-time version of the Grobman-Hartman Theorem). The additional conditions on the products are due to the fact that the dynamical system can define either an *orientation-preserving* or *orientation-reversing* map on the stable or unstable manifold near the fixed point. Recall that a diffeomorphism on $\mathbb{R}^l$ preserves orientation in $\mathbb{R}^l$ if $\det J > 0$, where J is its Jacobian matrix, and reverses it

otherwise. Two topologically equivalent maps must have the same orientation properties. The products in Theorem 2.4 are exactly the determinants of the Jacobian matrices of the map (2.12) *restricted* to its stable and unstable local invariant manifolds. It should be clear that one needs only account for *real* multipliers to compute these signs, since the product of a complex-conjugate pair of multipliers is always positive.

Let us consider two examples of fixed points.

Example 2.4 (Stable fixed points in $\mathbb{R}^1$) Suppose $x_0 = 0$ is a fixed point of a one-dimensional discrete-time system ($n = 1$). Let $n_- = 1$, meaning that the unique multiplier μ satisfies $|\mu| < 1$. In this case, according to Theorem 2.3, all orbits starting in some neighborhood of x_0 converge to x_0. Depending on the sign of the multiplier, we have the two possi-



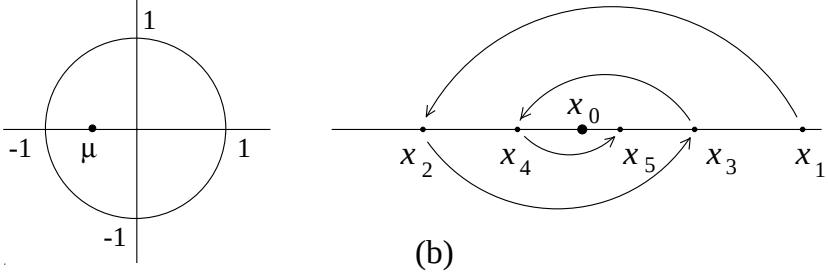


FIGURE 2.6. Stable fixed points of one-dimensional systems: (a) $0 < \mu < 1$; (b) $-1 < \mu < 0$.

bilities presented in Figure 2.6. If $0 < \mu < 1$, the iterations converge to x_0 *monotonously* (Figure 2.6(a)). If $-1 < \mu < 0$, the convergence is *non-monotonous* and the phase point "jumps" around x_0 while converging to x_0 (Figure 2.6(b)). In the first case the map preserves orientation in $\mathbb{R}^1$ while reversing it in the second. It should be clear that "jumping" orbits cannot be transformed into monotonous ones by a continuous map. Figure 2.7 presents orbits near the two types of fixed points using staircase diagrams. $\diamond$

Example 2.5 (Saddle fixed points in $\mathbb{R}^2$) Suppose $x_0 = 0$ is a fixed point of a two-dimensional discrete-time system (now $n = 2$). Assume that $n_- = n_+ = 1$, so that there is one (real) multiplier μ_1 outside the unit circle ($|\mu_1| > 1$) and one (real) multiplier μ_2 inside the unit circle ($|\mu_2| < 1$). In our case, there are two invariant manifolds passing through the fixed point, namely the one-dimensional manifold $W^s(x_0)$ formed by

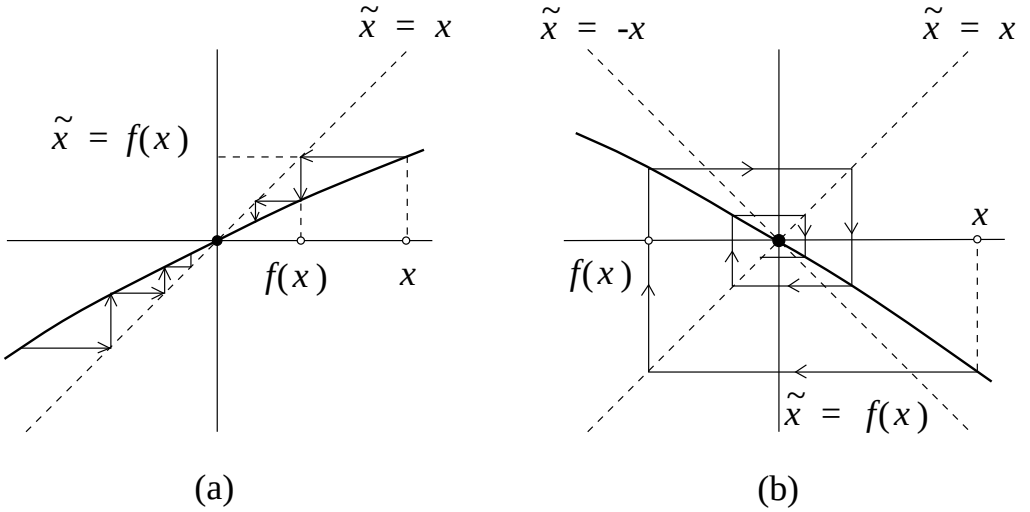


FIGURE 2.7. Staircase diagrams for stable fixed points.

orbits converging to x_0 under iterations of f, and the one-dimensional manifold $W^u(x_0)$ formed by orbits tending to x_0 under iterations of f^{-1}. Recall that the orbits of a discrete-time system are *sequences* of points. All orbits not belonging to the aforementioned manifolds pass near the fixed point and eventually leave its neighborhood in both "time" directions.

Figure 2.8 shows two types of saddles in $\mathbb{R}^2$. In the case (a) of positive multipliers, $0 < \mu_2 < 1 < \mu_1$, an orbit starting at a point on $W^s(x_0)$ converges to x_0 monotonously. Thus, the stable manifold $W^s(x_0)$ is formed by two invariant branches, $W^s_{1,2}(x_0)$, separated by x_0. The same can be

multipliers, $0 < \mu_2 < 1 < \mu_1$, an orbit starting at a point on $W^s(x_0)$ converges to x_0 monotonously. Thus, the stable manifold $W^s(x_0)$ is formed by two invariant branches, $W^s_{1,2}(x_0)$, separated by x_0. The same can be said about the unstable manifold $W^u(x_0)$ upon replacing f by its inverse. The restriction of the map onto both manifolds preserves orientation.

If the multipliers are negative (case (b)), $\mu_1 < -1 < \mu_2 < 0$, the orbits on the manifolds "jump" between the two components $W^{s,u}_{1,2}$ separated by x_0. The map reverses orientation in both manifolds. The branches $W^{s,u}_{1,2}$ are invariant with respect to the *second iterate* f^2 of the map. $\diamond$

Remarks:

(1) The stable and unstable manifolds $W^{s,u}(x_0)$ of a two-dimensional saddle are examples of *invariant curves*: If x belongs to the curve, so does any iterate $f^k(x)$. The invariant curve is not an orbit. Actually, it consists of an *infinite* number of orbits. Figure 2.9 shows invariant curves and an orbit near a saddle fixed point with positive multipliers.

(2) The global behavior of the stable and unstable manifolds $W^{s,u}(x_0)$ of a hyperbolic fixed point can be *very* complex, thus making the word "contain" absolutely necessary in Theorem 2.3.

Return, for example, to the planar case and suppose that x_0 is a saddle with positive multipliers. First of all, unlike the stable and unstable sets of an equilibrium in a continuous-time system, the manifolds $W^s(x_0)$ and $W^u(x_0)$ of a generic discrete-time system can *intersect* at nonzero angle (transversally) (see Figure 2.10(a)).

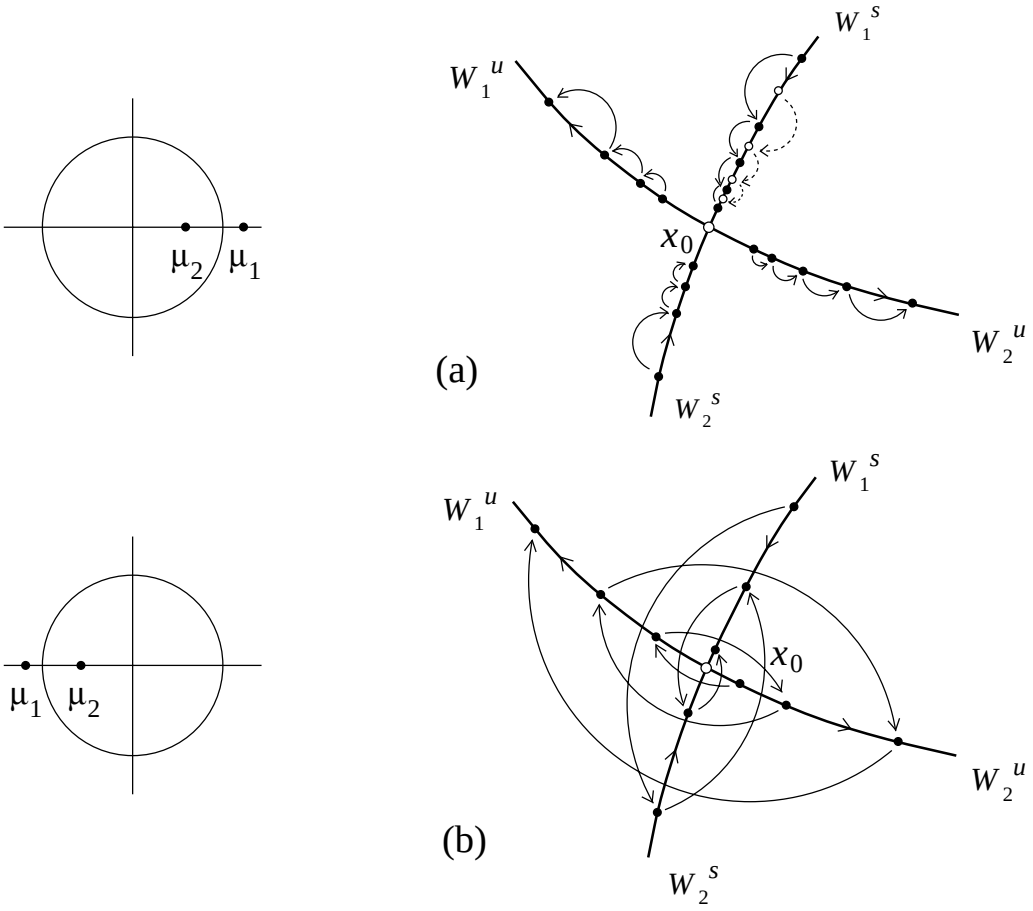


FIGURE 2.8. Invariant manifolds of saddle fixed points on the plane: (a) positive multipliers; (b) negative multipliers.

FIGURE 2.8. Invariant manifolds of saddle fixed points on the plane: (a) positive multipliers; (b) negative multipliers.

Moreover, one transversal intersection, if it occurs, implies an *infinite* number of such intersections. Indeed, let x^0 be a point of the intersection. By definition, it belongs to both invariant manifolds. Therefore, the orbit starting at this point converges to the saddle point x_0 under repeated iteration of either f or f^{-1} : $f^k(x^0) \to x_0$ as $k \to \pm\infty$. Each point of this orbit is a point of intersection of $W^s(x_0)$ and $W^u(x_0)$. This infinite number of intersections forces the manifolds to "oscillate" in a complex manner near x_0, as sketched in Figure 2.10(b). The resulting "web" is called the *Poincaré homoclinic structure*. The orbit starting at x^0 is said to be *homoclinic* to x_0. It is the presence of the homoclinic structure that can make the intersection of $W^{s,u}(x_0)$ with any neighborhood of the saddle x_0 highly nontrivial.

The dynamical consequences of the existence of the homoclinic structure are also dramatic: It results in the appearance of an *infinite number* of periodic points with arbitrary high periods near the homoclinic orbit. This follows from the presence of *Smale horseshoes* (see Chapter 1). Figure 2.11 illustrates how the horseshoes are formed. Take a (curvilinear) rectangle S near the stable manifold $W^s(x_0)$ and consider its iterations $f^k S$. If the homoclinic structure is present, for a sufficiently high number of iterations N, $f^N S$ will look like the folded and expanded band Q shown in the figure.

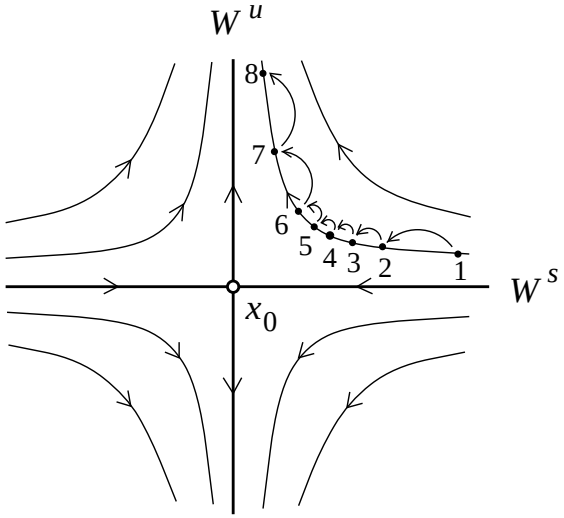
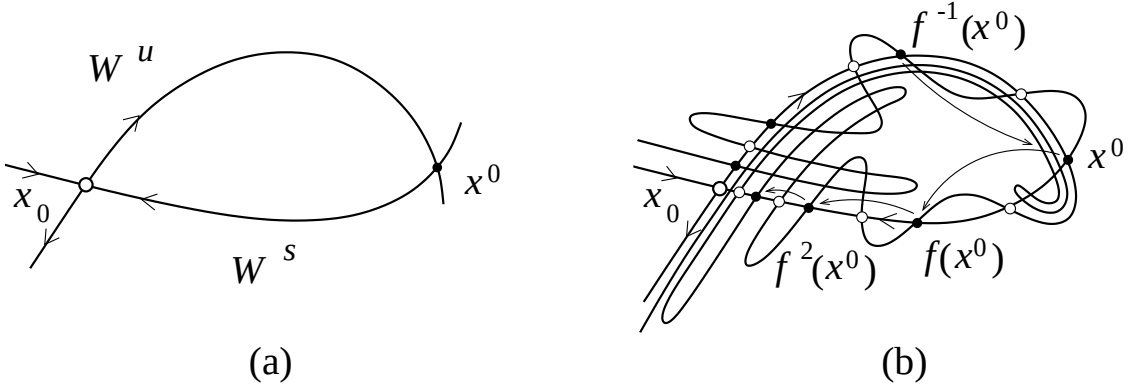


FIGURE 2.9. Invariant curves and an orbit near a saddle fixed point.



(a) (b)

FIGURE 2.10. Poincaré homoclinic structure.

The intersection of S with Q forms several horseshoes, where each of them implies an infinite number of cycles with arbitrary high periods. $\diamondsuit$

2.2.3 Hyperbolic limit cycles

Using the results of the previous section and the Poincaré map construction (see Chapter 1), we can define *hyperbolic limit cycles* in continuous-time systems and describe the topology of phase orbits near such cycles. Consider a continuous-time dynamical system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{2.13}$$

with smooth f, and assume that there is an isolated periodic orbit (limit cycle) L_0 of (2.13). As in Chapter 1, let Σ be a local cross-section to the cycle of dimension $(n-1)$ (codim $\Sigma = 1$) with coordinates $\xi = (\xi_1, \ldots, \xi_{n-1})^T$. System (2.13) locally defines a smooth invertible map P (a Poincaré map) from Σ to Σ along the orbits of (2.13). The point ξ_0 of intersection of L_0 with Σ is a fixed point of the map P, $P(\xi_0) = \xi_0$.

Generically, the fixed point ξ_0 is hyperbolic, so there exist invariant man-

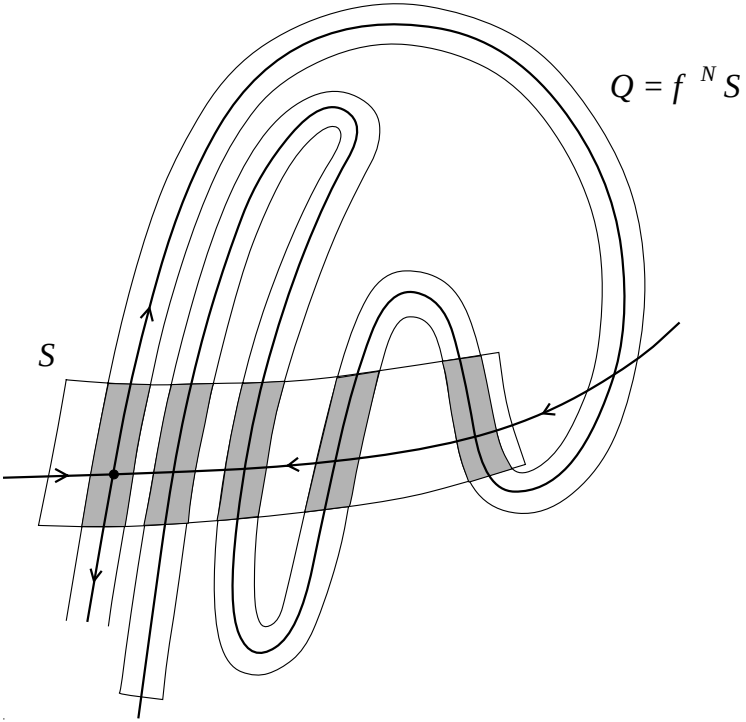


FIGURE 2.11. Smale horseshoes embedded into the Poincaré homoclinic structure.

ifolds

$$W^s(\xi_0) = \{\xi \in \Sigma : P^k(\xi) \to \xi_0,\ k \to +\infty\}$$

and

$$W^s(\xi_0) = \{\xi \in \Sigma : P^k(\xi) \to \xi_0,\ k \to +\infty\}$$

and

$$W^u(\xi_0) = \{\xi \in \Sigma : P^{-k}(\xi) \to \xi_0,\ k \to +\infty\},$$

of the dimensions n_- and n_+, respectively, where $n_\mp$ are the numbers of eigenvalues of the Jacobian matrix of P at ξ_0 located inside and outside the unit circle. Recall that $n_- + n_+ = n - 1$ and that the eigenvalues are called *multipliers of the cycle*. The invariant manifolds $W^{s,u}(\xi_0)$ are the intersections with Σ of the *stable and unstable manifolds of the cycle*:

$$\begin{aligned} W^s(L_0) &= \{x\ :\ \varphi^t x \to L_0,\ t \to +\infty\}, \\ W^u(L_0) &= \{x\ :\ \varphi^t x \to L_0,\ t \to -\infty\}, \end{aligned}$$

where φ^t is the flow corresponding to (2.13).

We can now use the results on the topological classification of fixed points of discrete-time dynamical systems to classify limit cycles. A limit cycle is called *hyperbolic* if ξ_0 is a hyperbolic fixed point of the Poincaré map. Similarly, a hyperbolic cycle is called a *saddle cycle* if it has multipliers both inside and outside the unit circle (i.e., $n_- n_+ \neq 0$). Recall that the product of the multipliers is always *positive* (see Chapter 1); therefore the Poincaré map preserves orientation in Σ. This imposes some restrictions on the location of the multipliers in the complex plane.

Example 2.6 (Hyperbolic cycles in planar systems) Consider a

smooth planar system

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2), \end{cases}$$

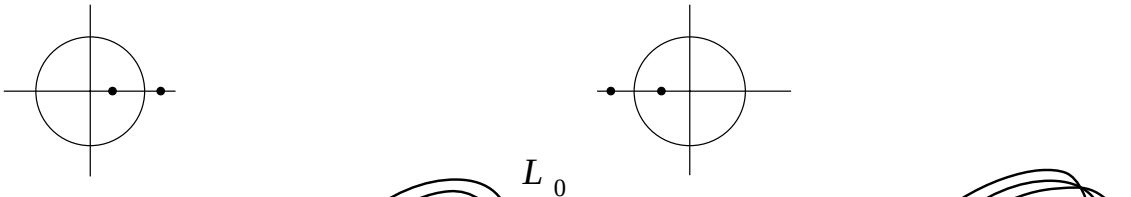
$x = (x_1, x_2)^T \in \mathbb{R}^2$. Let $x_0(t)$ be a solution corresponding to a limit cycle L_0 of the system, and let T_0 be the (minimal) period of this solution. There is only one multiplier of the cycle, μ_1, which is positive and is given by

$$\mu_1 = \exp\left\{\int_0^{T_0} (\text{div } f)(x_0(t))\ dt\right\} > 0,$$

where div stands for the *divergence* of the vector field:

$$(\text{div } f)(x) = \frac{\partial f_1(x)}{\partial x_1} + \frac{\partial f_2(x)}{\partial x_2}.$$

If $0 < \mu_1 < 1$, we have a stable hyperbolic cycle and all nearby orbits converge exponentially to it, while for $\mu_1 > 1$ we have an unstable hyperbolic cycle with exponentially diverging neighboring orbits. $\diamond$



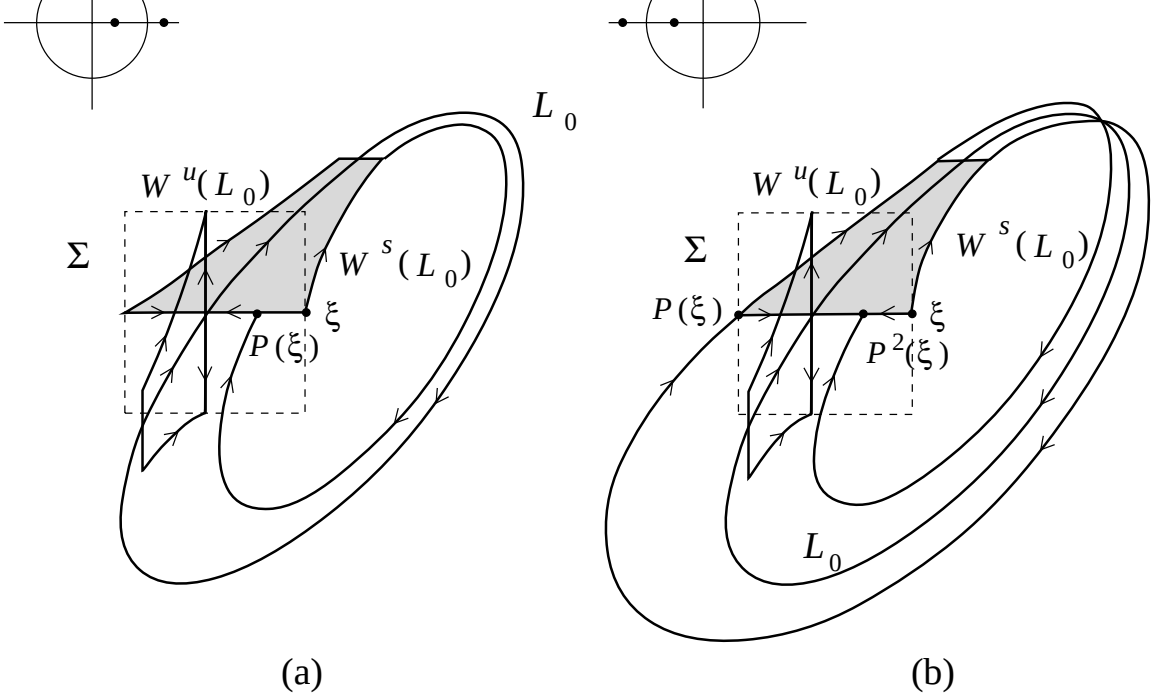


FIGURE 2.12. Saddle cycles in three-dimensional systems: (a) positive multipliers and (b) negative multipliers.

Example 2.7 (Saddle cycles in three-dimensional systems) Example 2.5 provides two types of saddle limit cycles existing in $\mathbb{R}^3$ (see Figure 2.12). If the multipliers of the Poincaré map satisfy

$$0 < \mu_2 < 1 < \mu_1,$$

both invariant manifolds $W^s(L_0)$ and $W^u(L_0)$ of the cycle L_0 are *simple* bands (Figure 2.12(a)), while in the case when the multipliers satisfy

$$\mu_1 < -1 < \mu_2 < 0,$$

the manifolds $W^s(L_0)$ and $W^u(L_0)$ are *twisted* bands (called *Möbius strips*) (see Figure 2.12(b)). Other types of saddle cycles in $\mathbb{R}^3$ are impossible, since the product of the multipliers of any Poincaré map is positive. Thus, the manifolds $W^s(L_0)$ and $W^u(L_0)$ must both be simple or twisted.

Finally, remark that $W^s(L)$ and $W^u(L)$ can *intersect* along orbits homoclinic to the cycle L, giving rise to Poincaré homoclinic structure and Smale horseshoes on the cross-section Σ. $\diamond$

2.3 Bifurcations and bifurcation diagrams

Now consider a dynamical system that depends on parameters. In the continuous-time case we will write it as

$$\dot{x} = f(x, \alpha), \tag{2.14}$$

while in the discrete-time case it is written as

$$x \mapsto f(x, \alpha) \tag{2.15}$$

while in the discrete-time case it is written as

$$x \mapsto f(x, \alpha), \tag{2.15}$$

where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^m$ represent phase variables and parameters, respectively. Consider the phase portrait of the system.[2] As the parameters vary, the phase portrait also varies. There are two possibilities: either the system remains topologically equivalent to the original one, or its topology changes.

Definition 2.11 *The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a* bifurcation.

Thus, a bifurcation is a change of the topological type of the system as its parameters pass through a *bifurcation* (*critical*) *value*. Actually, the central topic of this book is the classification and analysis of various bifurcations.

Example 2.8 (Andronov-Hopf bifurcation) Consider the following planar system that depends on one parameter:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 & = & x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2). \end{array} \right. \tag{2.16}$$

---

[2] If necessary, one may consider the phase portrait in a parameter-dependent region $U_\alpha \subset \mathbb{R}^n$.

In polar coordinates (ρ, θ) it takes the form

$$\begin{cases} \dot{\rho} &= \rho(\alpha - \rho^2), \\ \dot{\theta} &= 1, \end{cases} \tag{2.17}$$

and can be integrated explicitly (see Exercise 6). Since the equations for

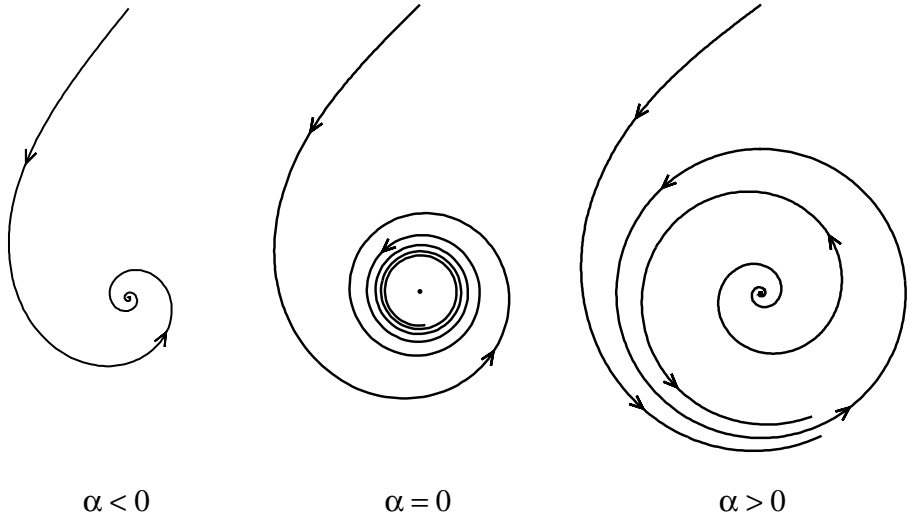


FIGURE 2.13. Hopf bifurcation.

ρ and θ are independent in (2.17), we can easily draw phase portraits of the system in a fixed neighborhood of the origin, which is obviously the only equilibrium point (see Figure 2.13). For $\alpha \leq 0$, the equilibrium is a

ρ and θ are independent in (2.17), we can easily draw phase portraits of the system in a fixed neighborhood of the origin, which is obviously the only equilibrium point (see Figure 2.13). For $\alpha \leq 0$, the equilibrium is a *stable focus*, since $\dot{\rho} < 0$ and $\rho(t) \to 0$, if we start from any initial point. On the other hand, if $\alpha > 0$, we have $\dot{\rho} > 0$ for small $\rho > 0$ (the equilibrium becomes an *unstable focus*), and $\dot{\rho} < 0$ for sufficiently large ρ. It is easy to see from (2.17) that the system has a *periodic orbit* for any $\alpha > 0$ of radius $\rho_0 = \sqrt{\alpha}$ (at $\rho = \rho_0$ we have $\dot{\rho} = 0$). Moreover, this periodic orbit is stable, since $\dot{\rho} > 0$ inside and $\dot{\rho} < 0$ outside the cycle.

Therefore, $\alpha = 0$ is a bifurcation parameter value. Indeed, a phase portrait with a limit cycle cannot be deformed by a one-to-one transformation into a phase portrait with only an equilibrium. The presence of a limit cycle is said to be a *topological invariant*. As α increases and crosses zero, we have a bifurcation in system (2.16) called the *Andronov-Hopf bifurcation*. It leads to the appearance, from the equilibrium state, of small-amplitude periodic oscillations. We will use this bifurcation as an example later in this chapter and analyze it in detail in Chapters 3 and 5. $\diamond$

As should be clear, an Andronov-Hopf bifurcation can be detected if we fix *any* small neighborhood of the equilibrium. Such bifurcations are called *local*. One can also define local bifurcations in discrete-time systems as those detectable in any small neighborhood of a fixed point. We will often refer to local bifurcations as *bifurcations of equilibria or fixed points*, although we will analyze not just these points but the whole phase portraits near the equilibria. Those bifurcations of limit cycles which correspond to

local bifurcations of associated Poincaré maps are called *local bifurcations of cycles*.

There are also bifurcations that cannot be detected by looking at small neighborhoods of equilibrium (fixed) points or cycles. Such bifurcations are called *global*.

Example 2.9 (Heteroclinic bifurcation) Consider the following planar system that depends on one parameter:

$$\begin{cases} \dot{x}_1 &= 1 - x_1^2 - \alpha x_1 x_2, \\ \dot{x}_2 &= x_1 x_2 + \alpha(1 - x_1^2). \end{cases} \tag{2.18}$$

The system has two saddle equilibria

$$x_{(1)} = (-1, 0), x_{(2)} = (1, 0),$$

for all values of α (see Figure 2.14). At $\alpha = 0$ the horizontal axis is invariant and, therefore, the saddles are connected by an orbit that is asymptotic to one of them for $t \to +\infty$ and to the other for $t \to -\infty$. Such orbits are called *heteroclinic*. Similarly, an orbit that is asymptotic to the same equilibrium as $t \to +\infty$ and $t \to -\infty$ is called *homoclinic*. For $\alpha \neq 0$, the x_1-axis is no longer invariant, and the connection disappears. This is obviously a global bifurcation. To detect this bifurcation we must fix a region U covering both saddles. We will study hetero- and homoclinic orbit bifurcations in Chapter 6. $\diamond$

bifurcation. To detect this bifurcation we must fix a region U covering both saddles. We will study hetero- and homoclinic orbit bifurcations in Chapter 6. $\diamond$

There are global bifurcations in which certain local bifurcations are involved. In such cases, looking at the local bifurcation provides only partial information on the behavior of the system. The following example illustrates this possibility.

Example 2.10 (Saddle-node homoclinic bifurcation) Let us analyze the following system on the plane:

$$\begin{cases} \dot{x}_1 &= x_1(1-x_1^2-x_2^2) \ - \ x_2(1+\alpha+x_1), \\ \dot{x}_2 &= x_1(1+\alpha+x_1) \ + \ x_2(1-x_1^2-x_2^2), \end{cases} \tag{2.19}$$

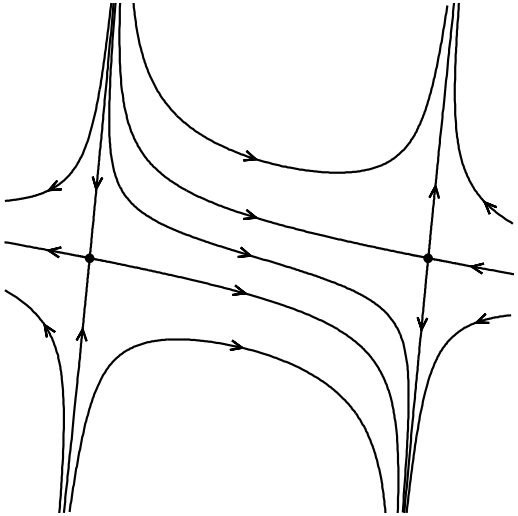
where α is a parameter. In polar coordinates (ρ, θ) system (2.19) takes the form

$$\begin{cases} \dot{\rho} &= \rho(1-\rho^2), \\ \dot{\theta} &= 1+\alpha+\rho\cos\theta. \end{cases} \tag{2.20}$$

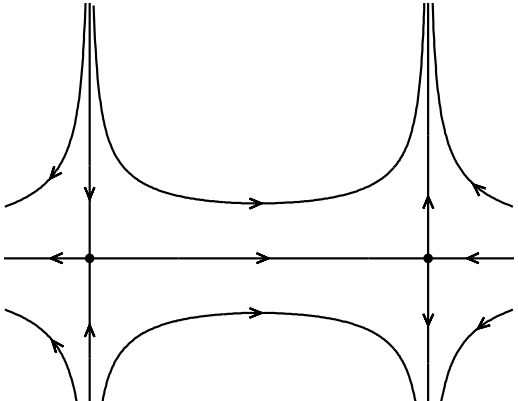
Fix a thin annulus U around the unit circle $\{(\rho,\theta) : \rho = 1\}$. At $\alpha = 0$, there is a nonhyperbolic equilibrium point of system (2.20) in the annulus:

$$x_0 = (\rho_0, \theta_0) = (1, \pi)$$

(see Figure 2.15). It has eigenvalues $\lambda_1 = 0, \lambda_2 = -2$ (check!). For small positive values of α the equilibrium disappears, while for small negative



$\alpha < 0$



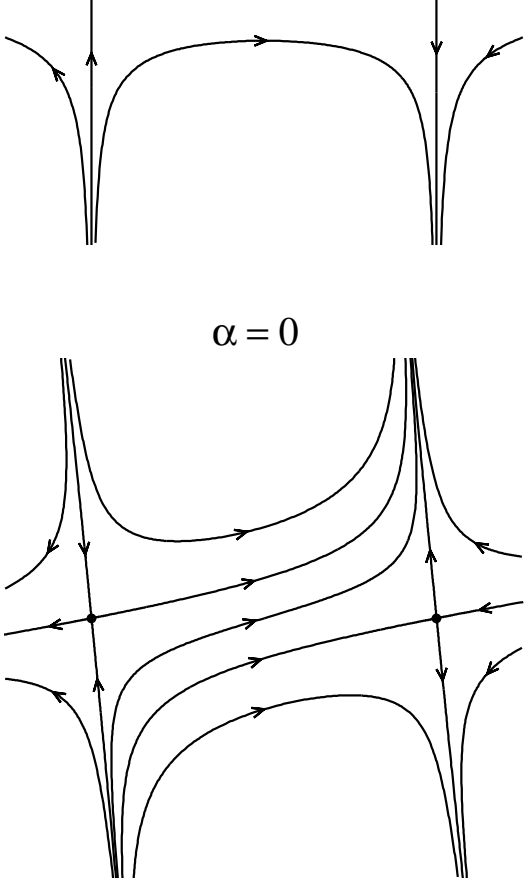


FIGURE 2.14. Heteroclinic bifurcation.

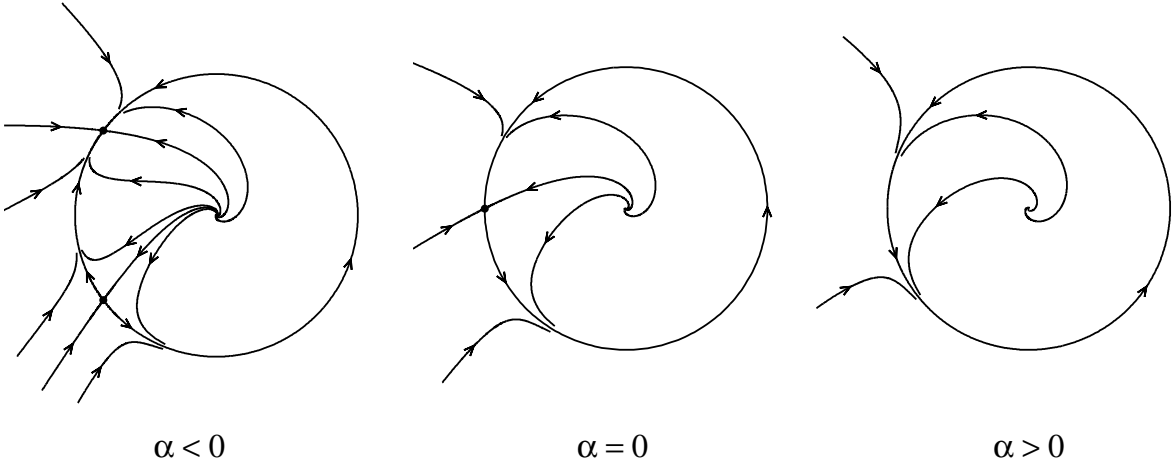


FIGURE 2.15. Saddle-node homoclinic bifurcation.

α it splits into a saddle and a node (this bifurcation is called a *saddle-node* or *fold* bifurcation; see Chapter 3). This is a local event. However, for $\alpha > 0$ a stable *limit cycle* appears in the system coinciding with the unit circle. This circle is always an invariant set in the system, but for $\alpha \leq 0$ it contains equilibria. Looking at only a small neighborhood of the nonhyperbolic equilibrium, we miss the global appearance of the cycle. Notice that at $\alpha = 0$ there is exactly one orbit that is homoclinic to the nonhyperbolic equilibrium x_0. We will discuss such global bifurcations in Chapter 7. $\diamond$

nonhyperbolic equilibrium x_0. We will discuss such global bifurcations in Chapter 7. $\diamond$

We return now to a general discussion of bifurcations in a parameter-dependent system (2.14) (or (2.15)). Take some value $\alpha = \alpha_0$ and consider a maximal connected parameter set (called a *stratum*) containing α_0 and composed by those points for which the system has a phase portrait that is topologically equivalent to that at α_0. Taking all such strata in the parameter space $\mathbb{R}^m$, we obtain the *parametric portrait* of the system. For example, system (2.16) exhibiting the Andronov-Hopf bifurcation has a parametric portrait with two strata: $\{\alpha \leq 0\}$ and $\{\alpha > 0\}$. In system (2.18) there are three strata: $\{\alpha < 0\}$, $\{\alpha = 0\}$, and $\{\alpha > 0\}$. Notice, however, that the phase portrait of (2.18) for $\alpha < 0$ is topologically equivalent to that for $\alpha > 0$.

The parametric portrait together with its characteristic phase portraits constitute a *bifurcation diagram.*

Definition 2.12 *A* bifurcation diagram *of the dynamical system is a stratification of its parameter space induced by the topological equivalence, together with representative phase portraits for each stratum.*

It is desirable to obtain the bifurcation diagram as a result of the qualitative analysis of a given dynamical system. It classifies in a very condensed way all possible modes of behavior of the system and transitions between

them (bifurcations) under parameter variations.[3] Note that the bifurcation diagram depends, in general, on the region of phase space considered.

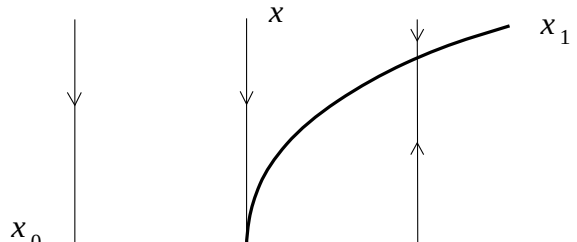
Remark:

If a dynamical system has a one- or two-dimensional phase space and depends on only one parameter, its bifurcation diagram can be visualized in the *direct product* of the phase and parameter spaces, $\mathbb{R}^{1,2} \times \mathbb{R}^1$ with the phase portraits represented by one- or two-dimensional *slices* $\alpha = \text{const}$.

Consider, for example, a scalar system

$$\dot{x} = \alpha x - x^3, \ x \in \mathbb{R}^1, \alpha \in \mathbb{R}^1.$$

This system has an equilibrium $x_0 = 0$ for all α. This equilibrium is stable for $\alpha < 0$ and unstable for $\alpha > 0$ (α is the eigenvalue of this equilibrium). For $\alpha > 0$, there are two extra equilibria branching from the origin (namely, $x_{1,2} = \pm\sqrt{\alpha}$) which are stable. This bifurcation is often called a *pitchfork bifurcation*, the reason for which becomes immediately clear if one has a look at the bifurcation diagram of the system presented in (x, α)-space (see Figure 2.16). Notice that the system demonstrating the pitchfork bi-



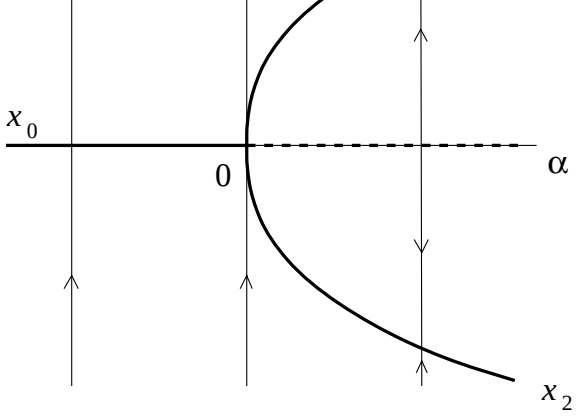


FIGURE 2.16. Pitchfork bifurcation.

furcation is *invariant* under the transformation $x \mapsto -x$. We will study bifurcations in such *symmetric* systems in Chapter 7. $\diamondsuit$

In the simplest cases, the parametric portrait is composed by a finite number of regions in $\mathbb{R}^m$. Inside each region the phase portrait is topologically equivalent. These regions are separated by *bifurcation boundaries*, which are smooth submanifolds in $\mathbb{R}^m$ (i.e., curves, surfaces). The boundaries can intersect, or meet. These intersections subdivide the boundaries into subregions, and so forth. A bifurcation boundary is defined by specifying a phase object (equilibrium, cycle, etc.) and some *bifurcation conditions*

---

[3]Recall that some time-related information on the behavior of the system is lost due to topological equivalence.

determining the type of its bifurcation (Hopf, fold, etc.). For example, the Andronov-Hopf bifurcation of an equilibrium is characterized by one bifurcation condition, namely, the presence of a purely imaginary pair of eigenvalues of the Jacobian matrix evaluated at this equilibrium (cf. Example 2.7):

$$\text{Re } \lambda_{1,2} = 0.$$

When a boundary is crossed, the bifurcation occurs.

Definition 2.13 *The* codimension *of a bifurcation in system* (2.14) *or* (2.15) *is the difference between the dimension of the parameter space and the dimension of the corresponding bifurcation boundary.*

Equivalently, the codimension (codim for short) is the number of independent conditions determining the bifurcation. This is the most practical definition of the codimension. It makes it clear that the codimension of a certain bifurcation is the same in all generic systems depending on a sufficient number of parameters.

Remark:

The bifurcation diagram of even a simple continuous-time system in a bounded region on the plane can be composed by an *infinite* number of strata. The situation becomes more involved for multidimensional continuous-time systems (with $n > 3$). In such systems the bifurcation values can be dense in some parameter regions and the parametric portrait

ber of strata. The situation becomes more involved for multidimensional continuous-time systems (with $n > 3$). In such systems the bifurcation values can be dense in some parameter regions and the parametric portrait can have a *Cantor* (*fractal*) structure with certain patterns repeated on smaller and smaller scales to infinity. Clearly, the task of fully investigating such a bifurcation diagram is practically impossible. Nevertheless, even partial knowledge of the bifurcation diagram provides important information about the behavior of the system being studied. $\diamondsuit$

2.4 Topological normal forms for bifurcations

Fortunately, bifurcation diagrams are not entirely "chaotic." Different strata of bifurcation diagrams in generic systems interact with each other following certain rules. This makes bifurcation diagrams of systems arising in many different applications look similar. To discuss this topic, we have to decide when two dynamical systems have "qualitatively similar" or equivalent bifurcation diagrams. Consider two (for definitiveness, continuous-time) dynamical systems:

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m, \tag{2.21}$$

and

$$\dot{y} = g(y, \beta), \quad y \in \mathbb{R}^n, \ \beta \in \mathbb{R}^m, \tag{2.22}$$

with smooth right-hand sides and the same number of variables and parameters. The following definition is parallel to Definition 2.1, with necessary modifications due to parameter dependence.

Definition 2.14 *Dynamical system* (2.21) *is called* topologically equivalent *to a dynamical system* (2.22) *if*

(i) *there exists a homeomorphism of the parameter space* $p : \mathbb{R}^m \to \mathbb{R}^m,\ \beta = p(\alpha)$;

(ii) *there is a parameter-dependent homeomorphism of the phase space* $h_\alpha : \mathbb{R}^n \to \mathbb{R}^n,\ y = h_\alpha(x)$, *mapping orbits of the system* (2.21) *at parameter values* α *onto orbits of the system* (2.22) *at parameter values* $\beta = p(\alpha)$, *preserving the direction of time.*

Clearly, the homeomorphism p transforms the parametric portrait of system (2.21) into the parametric portrait of system (2.22), while the homeomorphism h_α maps corresponding phase portraits. By definition, topologically equivalent parameter-dependent systems have (topologically) equivalent bifurcation diagrams.

Remark:

Notice that we do *not* require the homeomorphism h_α to depend *continuously* on α, which would imply that the map $(x, \alpha) \mapsto (h_{p(\alpha)}(x), p(\alpha))$ be a homeomorphism of the direct product $\mathbb{R}^n \times \mathbb{R}^m$. For this reason, some authors call the above-defined topological equivalence *weak* (or *fiber*) topological equivalence. $\diamondsuit$

a homeomorphism of the direct product $\mathbb{R}^n \times \mathbb{R}^m$. For this reason, some authors call the above-defined topological equivalence *weak* (or *fiber*) topological equivalence. $\diamondsuit$

As in the constant-parameter case, Definition 2.14 can be modified if one is interested in comparing local behavior of the systems, for example, in a small neighborhood of the origin of the state space, for small parameter values.

Definition 2.15 *Two systems* (2.21) *and* (2.22) *are called* locally topologically equivalent *near the origin, if there exists a map* $(x, \alpha) \mapsto (h_\alpha(x), p(\alpha))$, *defined in a small neighborhood of* $(x, \alpha) = (0, 0)$ *in the direct product* $\mathbb{R}^n \times \mathbb{R}^m$ *and such that*

(i) $p : \mathbb{R}^m \to \mathbb{R}^m$ *is a homeomorphism defined in a small neighborhood of* $\alpha = 0$, $p(0) = 0$;

(ii) $h_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ *is a parameter-dependent homeomorphism defined in a small neighborhood* U_α *of* $x = 0$, $h_0(0) = 0$, *and mapping orbits of* (2.21) *in* U_α *onto orbits of* (2.22) *in* $h_\alpha(U_\alpha)$, *preserving the direction of time.*

This definition means that one can introduce two small neighborhoods of the origin U_α and V_β, whose diameters are bounded away from zero uniformly for α, β varying in some fixed small neighborhoods of the origin of the corresponding parameter spaces. Then, the homeomorphism h_α maps orbits of (2.21) in U_α onto orbits of (2.22) in $V_{p(\alpha)}$, preserving their orientation.

We now consider the problem of the classification of all possible bifurcation diagrams of generic systems, at least, locally (i.e. near bifurcation boundaries in the parameter space and corresponding critical orbits in the phase space) and up to and including certain codimension. These local diagrams could then serve as "building blocks" to construct the "global" bifurcation diagram of any system. This problem has been solved for equilibrium bifurcations in two-dimensional continuous-time systems up to and including codim 3. In some sense, it has also been solved for bifurcations of equilibria and fixed points in multidimensional continuous- and discrete-time systems up to and including codim 2, although the relevant results are necessarily incomplete (see Chapters 3, 4, 8, and 9). There are also several outstanding results concerning higher-codimension local bifurcations and some global bifurcations of codim 1 and 2.

The classification problem formulated above is simplified due to the following obvious but important observation. The minimal number of free parameters required to meet a codim k bifurcation in a parameter-dependent system is exactly equal to k. Indeed, to satisfy a single bifurcation condition, we need, in general, to "tune" a (single) parameter of the system. If there are two conditions to be satisfied, two parameters have to be varied, and so forth. In other words, we have to control k parameters to reach a codim k bifurcation boundary in the parametric portrait of a generic system. On the other hand, it is enough to study a bifurcation of codim k in generic k-parameter systems. General m-parameter ($m > k$) diagrams near the bifurcation boundary can then be obtained by "shifting" the k-parameter

the other hand, it is enough to study a bifurcation of codim k in generic k-parameter systems. General m-parameter ($m > k$) diagrams near the bifurcation boundary can then be obtained by "shifting" the k-parameter diagram in the complementary directions. For example, the Andronov-Hopf bifurcation is a codim 1 (local) bifurcation. Thus, it occurs at isolated parameter values in systems depending on one parameter. In two-parameter systems, it generally occurs on specific curves (one-dimensional manifolds). If we cross this curve at a nonzero angle (transversally), the resulting one-parameter bifurcation diagrams (where the parameter, e.g., is the arclength along a transversal curve) will be topologically equivalent to the original one-parameter diagram. The same will be true if we cross a two-dimensional surface corresponding to the Hopf bifurcation in a system depending on three parameters.

For local bifurcations of equilibria and fixed points, universal bifurcation diagrams are provided by *topological normal forms*.[4] This is one of the central notions in bifurcation theory. Let us discuss it in the continuous-time setting, although it also applies to discrete-time systems. Sometimes it is possible to construct a simple (polynomial in ξ_i) system

$$\dot{\xi} = g(\xi, \beta; \sigma), \quad \xi \in \mathbb{R}^n, \ \beta \in \mathbb{R}^k, \ \sigma \in \mathbb{R}^l, \tag{2.23}$$

[4]It is possible to construct a kind of topological normal form for certain global bifurcations involving homoclinic orbits.

which has at $\beta = 0$ an equilibrium $\xi = 0$ satisfying k bifurcation conditions determining a codim k bifurcation of this equilibrium. Here σ is a vector of the coefficients $\sigma_i,\ i = 1, 2, \ldots, l$, of the polynomials involved in (2.23). In all the cases we will consider, there is a *finite* number of regions in the coefficient space corresponding to topologically nonequivalent bifurcation diagrams of (2.23). In the simplest situations, the σ_i take only a *finite* number of integer values. For example, all the coefficients $\sigma_i = 1$ except a single $\sigma_{i_0} = \pm 1$. In more complex situations, some components of σ may take real values (*modulae*).

Together with system (2.23), let us consider a system

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}^k, \tag{2.24}$$

having at $\alpha = 0$ an equilibrium $x = 0$.

Definition 2.16 (Topological normal form) *System* (2.23) *is called a* topological normal form *for the bifurcation if any generic system* (2.24) *with the equilibrium* $x = 0$ *satisfying the same bifurcation conditions at* $\alpha = 0$ *is locally topologically equivalent near the origin to* (2.23) *for some values of the coefficients* σ_i.

Of course, we have to explain what a *generic* system means. In all the cases we will consider, "generic" means that the system satisfies a finite number of *genericity conditions*. These conditions will have the form of nonequalities:

Of course, we have to explain what a *generic* system means. In all the cases we will consider, "generic" means that the system satisfies a finite number of *genericity conditions*. These conditions will have the form of nonequalities:

$$N_i[f] \neq 0, \ i = 1, 2, \ldots, s,$$

where each N_i is some (algebraic) function of certain partial derivatives of $f(x, \alpha)$ with respect to x and α evaluated at $(x, \alpha) = (0, 0)$. Thus, a "typical" parameter-dependent system satisfies these conditions. Actually, the value of σ is then determined by values of $N_i, \ i = 1, 2, \ldots, s$.

It is useful to distinguish those genericity conditions which are determined by the system at the critical parameter values $\alpha = 0$. These conditions can be expressed in terms of partial derivatives of $f(x, 0)$ with respect to x evaluated at $x = 0$, and are called *nondegeneracy conditions*. All the other conditions, in which the derivatives of $f(x, \alpha)$ with respect to the parameters α are involved, are called *transversality conditions*. The role of these two types of conditions is different. The nondegeneracy conditions guarantee that the critical equilibrium (*singularity*) is not too degenerate (i.e., typical in a class of equiliubria satisfying given bifurcation conditions), while the transversality conditions assure that the parameters "unfold" this singularity in a generic way.

If a topological normal form is constructed, its bifurcation diagram clearly has a universal meaning, since it immanently appears as a part of bifurcation diagrams of generic systems exhibiting the relevant bifurcation. System (2.16) from Example 2.7, by which we have illustrated the Andronov-Hopf

bifurcation, corresponds to the case $\sigma = -1$ in the two-dimensional *topological normal form* for this bifurcation:

$$\begin{cases} \dot{\xi}_1 &= \beta\xi_1 - \xi_2 + \sigma\xi_1(\xi_1^2 + \xi_2^2), \\ \dot{\xi}_2 &= \xi_1 + \beta\xi_2 + \sigma\xi_2(\xi_1^2 + \xi_2^2). \end{cases}$$

The conditions specifying generic systems that demonstrate this bifurcation are the following:

(H.1) $$\frac{d}{d\alpha}\text{Re } \lambda_{1,2}(\alpha)\Big|_{\alpha=0} \neq 0$$

and

(H.2) $$l_1(0) \neq 0.$$

The first condition (transversality) means that the pair of complex-conjugate eigenvalues $\lambda_{1,2}(\alpha)$ crosses the imaginary axis with nonzero speed. The second condition (nondegeneracy) implies that a certain combination of Taylor coefficients of the right-hand sides of the system (up to and including third-order coefficients) does not vanish. An explicit formula for $l_1(0)$ will be derived in Chapter 3, where we also prove that the above system is really a topological normal form for the Hopf bifurcation. We will also show that $\sigma = \text{sign } l_1(0)$.

Remark:

There is a closely related notion of *versal deformation* (or *universal un-*

$\sigma = \operatorname{sign} l_1(0)$.

Remark:

There is a closely related notion of *versal deformation* (or *universal unfolding*) for a bifurcation. First, we need to define what we mean by an *induced* system.

Definition 2.17 (Induced system) *The system*

$$\dot{y} = g(y, \beta), \quad y \in \mathbb{R}^n, \ \beta \in \mathbb{R}^m,$$

is said to be induced *by the system*

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m,$$

if $g(y, \beta) = f(y, p(\beta))$, *where* $p : \mathbb{R}^m \to \mathbb{R}^m$ *is a continuous map.*

Notice that the map p is not necessarily a homeomorphism, so it can be noninvertible.

Definition 2.18 (Versal deformation) *System* (2.23) *is a* versal deformation *for the corresponding local bifurcation if any system* (2.24), *with the equilibrium* $x = 0$ *satisfying the same bifurcation conditions and nondegeneracy conditions at* $\alpha = 0$, *is locally topologically equivalent near the origin to a system induced by* (2.23) *for some values of the coefficients* σ_i.

It can be proved, in many cases, that the topological normal forms we derive are actually versal deformations for the corresponding bifurcations (see also Exercise 7). $\diamondsuit$

2.5 Structural stability

There are dynamical systems whose phase portrait (in some domain) does not change qualitatively under all sufficiently small perturbations.

Example 2.11 (Persistence of a hyperbolic equilibrium) Suppose that x_0 is a hyperbolic equilibrium of a continuous-time system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{2.25}$$

where f is smooth, $f(x_0) = 0$. Consider, together with system (2.25), its one-parameter *perturbation*

$$\dot{x} = f(x) + \varepsilon g(x), \quad x \in \mathbb{R}^n, \tag{2.26}$$

where g is also smooth and ε is a small parameter; setting $\varepsilon = 0$ brings (2.26) back to (2.25). System (2.26) has an equilibrium $x(\varepsilon)$ for all sufficiently small $|\varepsilon|$ such that $x(0) = x_0$. Indeed, the equation defining equilibria of (2.26) can be written as

$$F(x, \varepsilon) = f(x) + \varepsilon g(x) = 0,$$

with $F(x_0, 0) = 0$. We also have $F_x(x_0, 0) = A_0$, where A_0 is the Jacobian matrix of (2.25) at the equilibrium x_0. Since $\det A_0 \neq 0$, because x_0 is hyperbolic, the Implicit Function Theorem guarantees the existence of a

with $F(x_0, 0) = 0$. We also have $F_x(x_0, 0) = A_0$, where A_0 is the Jacobian matrix of (2.25) at the equilibrium x_0. Since $\det A_0 \neq 0$, because x_0 is hyperbolic, the Implicit Function Theorem guarantees the existence of a smooth function $x = x(\varepsilon),\ x(0) = x_0$, satisfying

$$F(x(\varepsilon), \varepsilon) = 0$$

for small values of $|\varepsilon|$. The Jacobian matrix of $x(\varepsilon)$ in (2.26),

$$A_\varepsilon = \left(\frac{df(x)}{dx} + \varepsilon \frac{dg(x)}{dx} \right) \Bigg|_{x=x(\varepsilon)},$$

depends smoothly on ε and coincides with A_0 in (2.25) at $\varepsilon = 0$. As already known, the eigenvalues of a matrix that depends smoothly on a parameter change *continuously* with the variation of this parameter.[5] Therefore, $x(\varepsilon)$ will have no eigenvalues on the imaginary axis for all sufficiently small $|\varepsilon|$, since it has no such eigenvalues at $\varepsilon = 0$. In other words, $x(\varepsilon)$ is a hyperbolic equilibrium of (2.26) for all $|\varepsilon|$ small enough. Moreover, the numbers n_- and n_+ of the stable and unstable eigenvalues of A_ε are fixed for these values of ε. Applying Theorem 2.2, we find that systems (2.25) and (2.26) are locally topologically equivalent near the equilibria. Actually, for every $|\varepsilon|$ small, there is a neighborhood $U_\varepsilon \subset \mathbb{R}^n$ of the equilibrium x_ε in which system (2.26) is topologically equivalent to (2.25) in U_0. In short,

---

[5] The eigenvalues vary smoothly as long as they remain *simple*.

all these facts are summarized by saying that "a hyperbolic equilibrium is *structurally stable* under smooth perturbations."

Similar arguments provide the persistence of a hyperbolic equilibrium for all sufficiently small $|\varepsilon|$ in a smooth system

$$\dot{x} = G(x, \varepsilon), \quad x \in \mathbb{R}^n, \ \varepsilon \in \mathbb{R}^1,$$

where $G(x, 0) = f(x)$. $\diamond$

The parameter ε from Example 2.11 somehow measures the distance between system (2.25) and its perturbation (2.26); if $\varepsilon = 0$ the systems coincide. There is a general definition of the distance between two smooth dynamical systems. Consider two continuous-time systems

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{2.27}$$

and

$$\dot{x} = g(x), \quad x \in \mathbb{R}^n, \tag{2.28}$$

with smooth f and g.

Definition 2.19 *The* distance *between* (2.27) *and* (2.28) *in a closed region* $U \subset \mathbb{R}^n$ *is a positive number* d_1 *given by*

$$d_1 = \sup_{x \in U} \left\{ \|f(x) - g(x)\| + \left\| \frac{df(x)}{dx} - \frac{dg(x)}{dx} \right\| \right\}.$$

$$d_1 = \sup_{x \in U} \left\{ \|f(x) - g(x)\| + \left\| \frac{df(x)}{dx} - \frac{dg(x)}{dx} \right\| \right\}.$$

The systems are ε-close *in* U *if* $d_1 \leq \varepsilon$.

Here $\|\cdot\|$ means a vector and a matrix norm in $\mathbb{R}^n$, for example:

$$\|x\| = \sqrt{\sum_{i=1,\ldots,n} x_i^2}, \quad \|A\| = \sqrt{\sum_{i,j=1,\ldots,n} a_{ij}^2}.$$

Thus, two systems are close if their right-hand sides are close to each other, *together* with their first partial derivatives. In this case one usually calls the systems C^1-*close*. Clearly, the distance between systems (2.25) and (2.26) is proportional to $|\varepsilon|$: $d_1 = C|\varepsilon|$ for some constant $C > 0$ depending on the upper bounds for $\|g\|$ and $\left\| \frac{dg}{dx} \right\|$ in U. Definition 2.19 can be applied verbatim to discrete-time systems.

Remark:

The appearance of the first derivatives in the definition of the distance is natural if one wants to ensure that close systems have nearby equilibria of the same topological type (see Example 2.11). It is easy to construct a smooth system (2.28) that is ε-close to (2.27) in the C^0-*distance*:

$$d_0 = \sup_{x \in U} \{\|f(x) - g(x)\|\},$$

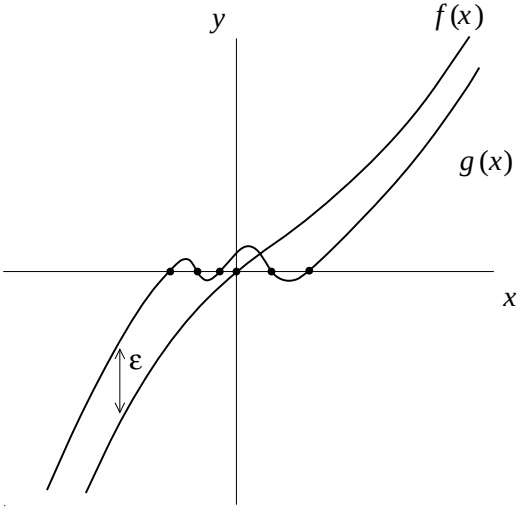


FIGURE 2.17. Two C^0-close functions with different numbers of zeros.

and that has a totally different number of equilibria in any neighborhood of an equilibrium of (2.27) (see Figure 2.17 for $n = 1$). $\diamondsuit$

Now we would like to define a structurally stable system, which means that any sufficiently close system is topologically equivalent to the structurally stable one. The following definition seems rather natural.

Definition 2.20 (Strict structural stability) *System* (2.27) *is* strictly structurally stable *in the region* U *if any system* (2.28) *that is sufficiently* C^1*-close in* U *is topologically equivalent in* U *to* (2.27).

Definition 2.20 (Strict structural stability) *System* (2.27) *is* strictly structurally stable *in the region U if any system* (2.28) *that is sufficiently C^1-close in U is topologically equivalent in U to* (2.27).

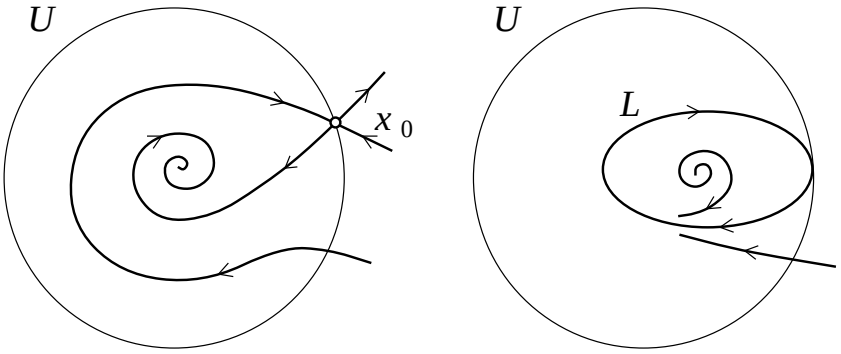


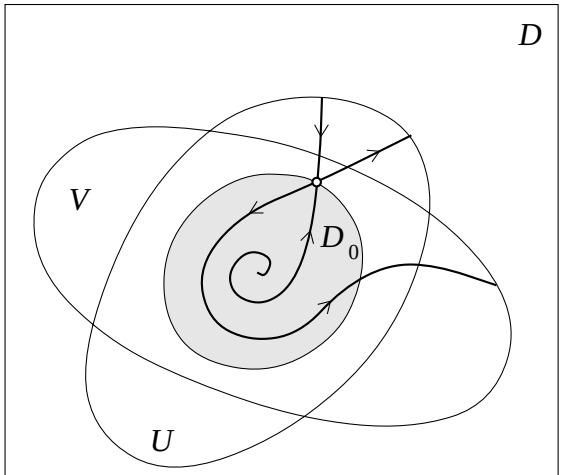
FIGURE 2.18. Structurally unstable orbits according to Definition 2.20.

Notice, however, that systems having *hyperbolic* equilibria on the boundary of U, or *hyperbolic* cycles touching the boundary (see Figure 2.18), are structurally *unstable* in accordance with this definition, since there are small system perturbations moving such equilibria out of U, or pushing such cycles to lie (partially) outside of U. There are two ways to handle this difficulty.

The first is to consider dynamical systems "in the whole phase space" and to forget about any regions. This way is perfect for dynamical systems defined on a *compact* smooth manifold X. In such a case, the "region U" in Definition 2.20 (as well as in the definition of the distance) should be

substituted by the "compact manifold X." Unfortunately, for systems in $\mathbb{R}^n$ this easily leads to complications. For example, the distance between many innocently looking systems may be *infinite* if the supremum in d_1 is taken over the whole of $\mathbb{R}^n$. Therefore, the second way out is to continue to work with bounded regions but to introduce another definition of structural stability.

Definition 2.21 (Andronov's structural stability) *A system* (2.27) *defined in a region $D \subset \mathbb{R}^n$ is called* structurally stable *in a region $D_0 \subset D$ if for any sufficiently C^1-close in D system* (2.28) *there are regions $U, V \subset D$, $D_0 \subset U$ such that* (2.27) *is topologically equivalent in U to* (2.28) *in V* (*see* Figure 2.19).



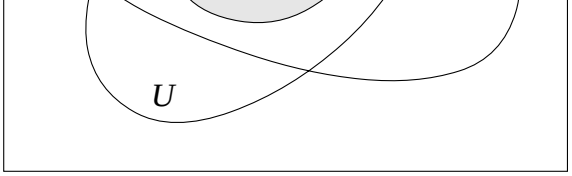


FIGURE 2.19. Andronov's structural stability.

A parallel definition can be given for discrete-time systems. If (2.27) is structurally stable in $D_0 \subset D$, then it is structurally stable in any region $D_1 \subset D_0$. There are cases when Definitions 2.20 and 2.21 actually coincide.

Lemma 2.1 *If a system is structurally stable in a region D_0 with the boundary B_0 and all its orbits point strictly inside B_0, then it is strictly structurally stable in $U = D_0$.* □

The following classical theorem gives necessary and sufficient conditions for a continuous-time system in the plane to be structurally stable.

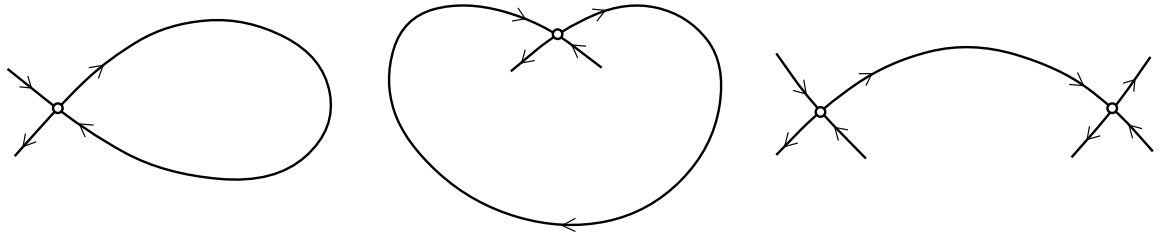


FIGURE 2.20. Structurally unstable connecting orbits in planar systems.

Theorem 2.5 (Andronov & Pontryagin [1937]) *A smooth dynamical system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2,$$

is structurally stable in a region $D_0 \subset \mathbb{R}^2$ if and only if

(i) *it has a finite number of equilibria and limit cycles in D_0, and all of them are hyperbolic*;

(ii) *there are no saddle separatrices returning to the same saddle or connecting two different saddles in D_0* (*see* Figure 2.20). □

Remark:

Actually, in their original paper of 1937, Andronov and Pontryagin considered systems with *analytic* right-hand sides in a region $D_0 \subset \mathbb{R}^2$ bounded by a (piecewise) smooth curve. They also assumed that all orbits point strictly inside the region, so they were able to use Definition 2.20. Later, Definition 2.21 was introduced and this restriction on the behavior on the boundary was left out. Moreover, they proved that the homeomorphism h transforming the phase portrait of a perturbed system in D_0 into that of the original system can be selected C^0-close to the identity map $\mathrm{id}(x) = x$. ♢

This theorem gives the complete description of structurally stable systems on the plane. It is rather obvious, although it has to be proved, that a typical (generic) system on the plane satisfies Andronov-Pontryagin con-

This theorem gives the complete description of structurally stable systems on the plane. It is rather obvious, although it has to be proved, that a typical (generic) system on the plane satisfies Andronov-Pontryagin conditions and is, thus, structurally stable. If one considers the bifurcation diagram of a generic planar system depending on k parameters, these are structurally stable systems that occupy k-dimensional open *regions* in the parameter space.

One can ask if a similar theorem exists for n-dimensional systems. The answer is "no." More precisely, one can establish *sufficient* conditions (called *Morse-Smale conditions*, similar to those in Theorem 2.5) for a continuous-time system to be structurally stable. Nevertheless, there are systems, which do not satisfy these conditions, that are structurally stable. In particular, structurally stable systems can have an *infinite* number of periodic orbits in compact regions. To understand this phenomenon, consider a continuous-time system $\mathbb{R}^3$. Suppose that there is a two-dimensional cross-section Σ on which the system defines a Poincaré map generating a *Smale horseshoe* (see Chapter 1 and Example 2.7 in this chapter). Then, the system has an infinite number of saddle cycles in some region of the phase space. A C^1-close system will define a C^1-close Poincaré map on Σ. The horseshoe will be slightly deformed, but the geometrical construction we have carried out in Chapter 1 remains valid. Thus, a complex invariant set including an infinite number of saddle cycles will persist under all sufficiently small perturbations. A homeomorphism transforming the corresponding phase portraits can also be constructed.

Moreover, it is possible to construct a system that has no close structurally stable systems. We direct the reader to the literature cited in this chapter's appendix.

2.6 Exercises

(1) Determine which of the following linear systems has a structurally stable equilibrium at the origin, and sketch its phase portrait:

(a)
$$\begin{cases} \dot{x} &= x - 2y, \\ \dot{y} &= -2x + 4y; \end{cases}$$

(b)
$$\begin{cases} \dot{x} &= 2x + y, \\ \dot{y} &= -x; \end{cases}$$

(c)
$$\begin{cases} \dot{x} &= x + 2y, \\ \dot{y} &= -x - y. \end{cases}$$

(2) The following system of partial differential equations is the FitzHugh-Nagumo caricature of the Hodgkin-Huxley equations modeling the nerve

(2) The following system of partial differential equations is the FitzHugh-Nagumo caricature of the Hodgkin-Huxley equations modeling the nerve impulse propagation along an axon:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - f_a(u) - v, \\ \frac{\partial v}{\partial t} &= bu, \end{aligned}$$

where $u = u(x,t)$ represents the membrane potential, $v = v(x,t)$ is a "recovery" variable, $f_a(u) = u(u-a)(u-1), 1 > a > 0, b > 0, -\infty < x < +\infty$, and $t > 0$.

Traveling waves are solutions to these equations of the form

$$u(x,t) = U(\xi),\ v(x,t) = V(\xi),\ \xi = x + ct,$$

where c is an a priori unknown wave propagation speed. The functions $U(\xi)$ and $V(\xi)$ are the wave profiles.

(a) Derive a system of three ordinary differential equations for the profiles with "time" ξ. (*Hint:* Introduce an extra variable: $W = \dot{U}$.)

(b) Check that for all $c > 0$ the system for the profiles (*the wave system*) has a unique equilibrium with one positive eigenvalue and two eigenvalues with negative real parts. (*Hint:* First, verify this assuming that eigenvalues are real. Then, show that the characteristic equation cannot have roots

on the imaginary axis, and finally, use the continuous dependence of the eigenvalues on the parameters.)

(c) Conclude that the equilibrium can be either a saddle or a saddle-focus with a one-dimensional unstable and a two-dimensional stable invariant manifold, and find a condition on the system parameters that defines a boundary between these two cases. Plot several boundaries in the (a, c)-plane for different values of b and specify the region corresponding to saddle-foci. (*Hint:* At the boundary the characteristic polynomial $h(\lambda)$ has a double root $\lambda_0 : h(\lambda_0) = h'(\lambda_0) = 0$.)

(d) Sketch possible profiles of traveling *impulses* in both regions. (*Hint:* An impulse corresponds to a solution of the wave system with

$$(U(\xi), V(\xi), W(\xi)) \to (0, 0, 0)$$

as $\xi \to \pm\infty$. See Chapter 6 for further details.)

(3) Prove that the system

$$\begin{cases} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -x_2, \end{cases}$$

is locally topologically equivalent near the origin to the system

$$\begin{cases} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -2x_2. \end{cases}$$

$$\begin{cases} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -2x_2. \end{cases}$$

(*Hint:* Mimic the proof of Example 2.1 without introducing polar coordinates.) Are the systems diffeomorphic?

(4) (Diffeomorphic limit cycles) Show that for diffeomorphic continuous-time systems, corresponding limit cycles have coinciding periods and multipliers. (*Hint:* Use the fact that variational equations around corresponding cycles (considered as autonomous systems with an extra cyclic variable) are diffeomorphic.)

(5) (Orbital equivalence and global flows)

(a) Prove that the scalar system

$$\frac{dx}{dt} = x^2, \quad x \in \mathbb{R}^1,$$

having solutions approaching infinity within finite time, and thus defining only *local* flow $\varphi^t : \mathbb{R}^1 \to \mathbb{R}^1$, is orbitally equivalent to the scalar system

$$\frac{dx}{d\tau} = \frac{x^2}{1+x^2}, \quad x \in \mathbb{R}^1,$$

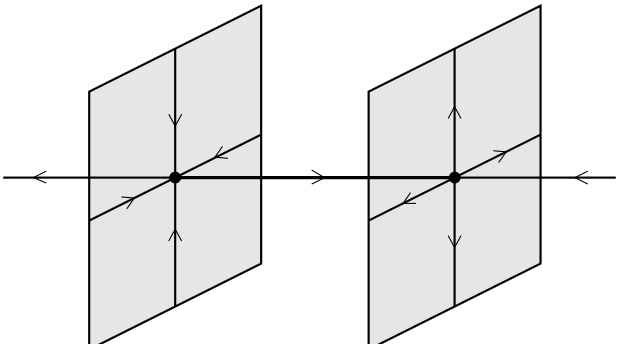
having no such solutions and therefore defining a *global* flow $\psi^\tau : \mathbb{R}^1 \to \mathbb{R}^1$. How are t and τ related?

(b) Prove that any smooth system $\dot{x} = f(x),\ x \in \mathbb{R}^n$, is orbitally equivalent in $\mathbb{R}^n$ to a smooth system defining a global flow ψ^τ on $\mathbb{R}^n$. (*Hint:* The system

$$\dot{x} = \frac{1}{1 + \|f(x)\|} f(x),$$

where $\|\cdot\|$ is the norm associated with the standard scalar product in $\mathbb{R}^n$, does the job.)

(6) (One-point parametric portrait) Construct an autonomous system of differential equations in $\mathbb{R}^3$ depending on two parameters (α, β) and having topologically equivalent phase portraits for all parameter values *except* $(\alpha, \beta) = (0, 0)$. (*Hint:* Use the idea of Example 2.9. At $\alpha = \beta = 0$, the system should have two saddle points with one-dimensional unstable and one-dimensional stable manifolds with coinciding branches (see Figure 2.21).)



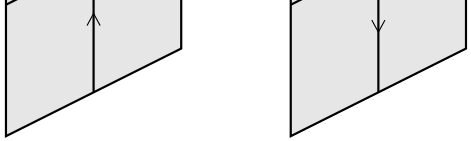


FIGURE 2.21. Exercise 6.

(7) (Induced systems) Show that the scalar system

$$\dot{y} = \beta y - y^2,$$

which exhibits the *transcritical* bifurcation, is topologically equivalent (in fact, diffeomorphic) to a system induced by the system

$$\dot{x} = \alpha - x^2,$$

which undergoes the fold bifurcation. (*Hint:* See Arrowsmith & Place [1990, p.193].)

(8) (Proof of Lemma 2.1)

(a) Prove that a smooth planar system $\dot{x} = f(x),\ x \in \mathbb{R}^2$, is topologically equivalent (in fact, diffeomorphic) in a region U, that is, free of equilibria and periodic orbits and is bounded by two orbits and two smooth curves transversal to orbits, to the system

$$\begin{cases} \dot{y}_1 &= 1, \\ \dot{y}_2 &= 0, \end{cases}$$

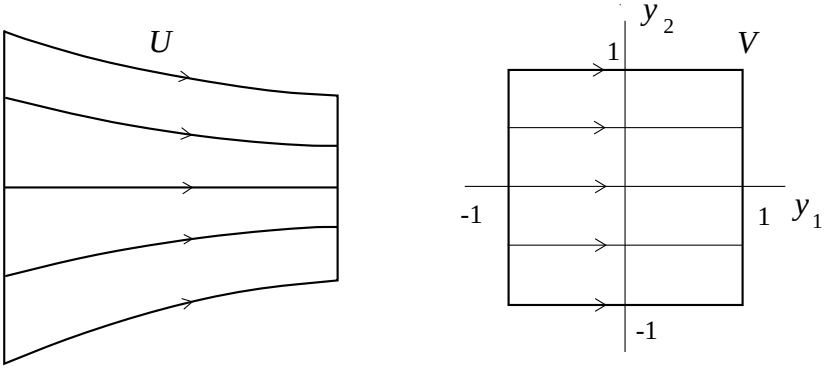


FIGURE 2.22. Phase portraits in regions U and V are equivalent.

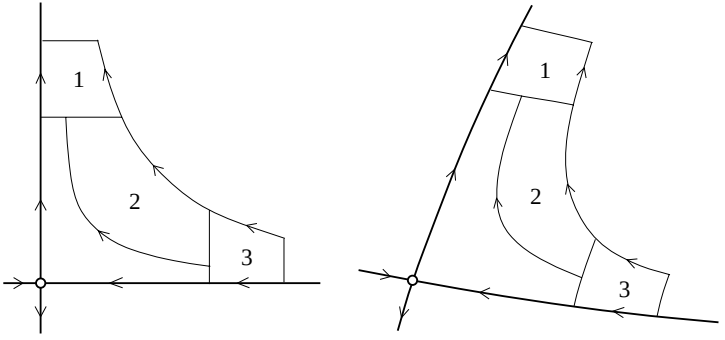


FIGURE 2.23. Saddles are topologically equivalent.

in the unit square $V = \{(y_1, y_2) : |y_1| \leq 1,\ |y_2| \leq 1\}$ (see Figure 2.22).

(b) Generalize this result to n-dimensional systems and prove Lemma

in the unit square $V = \{(y_1, y_2) : |y_1| \leq 1,\ |y_2| \leq 1\}$ (see Figure 2.22).

(b) Generalize this result to n-dimensional systems and prove Lemma 2.1.

(c) Prove, using part (a), that two hyperbolic saddle points on the plane have locally topologically equivalent phase portraits. (*Hint:* See Figure 2.23; an explicit map providing the equivalence is constructed in Chapter 6.) Where is the differentiability lost?

2.7 Appendix: Bibliographical notes

The notion of topological equivalence of dynamical systems appeared in the paper by Andronov & Pontryagin [1937] devoted to structurally stable systems on the plane. It is extensively used (among other equivalences) in singularity theory to classify singularities of maps and their deformations (Thom [1972], Arnold, Varchenko & Guseyn-Zade [1985], Golubitsky & Schaeffer [1985]).

The local topological equivalence of a nonlinear dynamical system to its linearization at a hyperbolic equilibrium was proved by Grobman [1959] and Hartman [1963]. See Hartman [1964] for details. Local topological equivalence of a map near a hyperbolic fixed point to its linearization has been established by Grobman and Hartman as a by-product of their proofs of the corresponding theorem in the continuous-time case (see also Nitecki

[1971]). A constructive proof of the topological equivalence of two linear systems with $n_0 = 0$ and the same n_- and n_+ can be found in Arnold [1973] and Hale & Koçak [1991].

The Local Stable Manifold Theorem for differential equations originated in works by Hadamard [1901] and Perron [1930]. Complete proofs and generalizations are given by Kelley [1967]; Hirsch, Pugh & Shub [1977] (see also Irwin [1980]). The Local Stable Manifold Theorem for maps is actually the main technical tool used to prove the relevant theorem for differential equations. Therefore, its proof can be found in the cited literature, for example, in Hartman [1964] or Nitecki [1971]. The latter reference also contains a proof that the stable and unstable sets of a hyperbolic fixed point are images of $\mathbb{R}^n_-$ and $\mathbb{R}^n_+$ under immersion.

The complex structure generated by the transversal intersection of the stable and unstable manifolds of a hyperbolic fixed point was discovered by Poincaré [1892,1893,1899] while analyzing area-preserving (conservative) maps appearing in celestial mechanics. Further analysis of this phenomenon in the conservative case was undertaken by Birkhoff [1935], with particular emphasis to the statistical properties of corresponding orbits. The nonconservative case was studied by Smale [1963], Neimark [1967], and Shil'nikov [1967*b*]. A nice exposition of this topic is given by Moser [1973].

There are two main approaches to studying bifurcations in dynamical systems. The first one, originating in the works by Poincaré, is to analyze the appearance (*branching*) of new phase objects of a certain type (equi-

There are two main approaches to studying bifurcations in dynamical systems. The first one, originating in the works by Poincaré, is to analyze the appearance (*branching*) of new phase objects of a certain type (equilibria or cycles, for example) from some known ones when parameters of the system vary. This approach led to the development of branching theory for equilibrium solutions of finite- and infinite-dimensional nonlinear equations (see, e.g, Vaĭnberg & Trenogin [1974], and Chow & Hale [1982]). The approach also proved to be a powerful tool to study some global bifurcations (see the bibliographical notes in Chapter 6). The second approach, going back to Andronov [1933] and reintroduced by Thom [1972] in order to classify gradient systems $\dot{x} = -\text{grad}\, V(x, \alpha)$, is to study rearrangements (*bifurcations*) of the whole phase portrait under variations of parameters. In principle, the branching analysis should precede more complete phase portrait study, but there are many cases where complete phase portraits are unavailable and studying certain solutions is the only way to get some information on the bifurcation.

Bifurcations of phase portraits of two-dimensional dynamical systems have been studied in great detail by Andronov and his co-workers in 1930-1950 and summarized in the classical book whose English translation is available as Andronov, Leontovich, Gordon & Maier [1973]. In his famous lectures, Arnold [1972] first applied many ideas from singularity theory of differentiable maps to dynamical systems (a similar approach was developed by Takens [1974*a*]). The notions of topological equivalence of parameter-dependent systems (*families*), versal deformations for local bi-

furcations, as well as many original results, were first presented in Arnold's lectures and then in the book by Arnold [1983]. Notice that in the literature in English versal deformations are often called *universal unfoldings* following terminology from singularity theory. A fundamental survey of bifurcation theory, including results on global bifurcations, is given by Arnol'd, Afraimovich, Il'yashenko & Shil'nikov [1994].

Structurally stable two-dimensional ODE systems were studied by Andronov & Pontryagin [1937] under the name *coarse* (or *rough*) systems. Actually, they included the requirement that the homeomorphism transforming the phase portraits be close to the identity. Peixoto [1962] proved that a typical system on a two-dimensional manifold is structurally stable. To discuss "typicality" one has to specify a space $\mathcal{D}$ of considered dynamical systems. Then, a property is called *typical* (or *generic*) if systems from the intersection of a countable number of open and dense subsets of $\mathcal{D}$ possess this property (see Wiggins [1990] for an introductory discussion). A class of structurally stable, multidimensional dynamical systems (called *Morse-Smale systems*) has been identified Smale [1961, 1967]. Such systems have only a finite number of equilibria and cycles, all of which are hyperbolic and have their stable and unstable invariant manifolds intersecting at nonzero angles (*transversally*). There are structurally stable systems that do not satisfy Morse-Smale criteria, in particular, having an *infinite* number of hyperbolic cycles [Smale 1963]. Moreover, structural stability is *not* a typical property for multidimensional dynamical systems, and structurally stable systems are not dense in a space $\mathcal{D}$ of smooth dynamical systems

ber of hyperbolic cycles [Smale 1963]. Moreover, structural stability is *not* a typical property for multidimensional dynamical systems, and structurally stable systems are not dense in a space $\mathcal{D}$ of smooth dynamical systems [Smale 1966]. The interested reader is referred to Nitecki [1971] and Arnold [1983] for more information.

3

One-Parameter Bifurcations of Equilibria in Continuous-Time Dynamical Systems

In this chapter we formulate conditions defining the simplest bifurcations of equilibria in n-dimensional continuous-time systems: the fold and the Hopf bifurcations. Then we study these bifurcations in the lowest possible dimensions: the fold bifurcation for scalar systems and the Hopf bifurca-

of equilibria in n-dimensional continuous-time systems: the fold and the Hopf bifurcations. Then we study these bifurcations in the lowest possible dimensions: the fold bifurcation for scalar systems and the Hopf bifurcation for planar systems. Chapter 5 shows how to "lift" these results to n-dimensional situations.

3.1 Simplest bifurcation conditions

Consider a continuous-time system depending on a parameter

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^1,$$

where f is smooth with respect to both x and α. Let $x = x_0$ be a hyperbolic equilibrium in the system for $\alpha = \alpha_0$. As we have seen in Chapter 2, under a small parameter variation the equilibrium moves slightly but remains hyperbolic. Therefore, we can vary the parameter further and monitor the equilibrium. It is clear that there are, generically, only two ways in which the hyperbolicity condition can be violated. Either a simple real eigenvalue approaches zero and we have $\lambda_1 = 0$ (see Figure 3.1(a)), or a pair of simple complex eigenvalues reaches the imaginary axis and we have $\lambda_{1,2} = \pm i\omega_0, \ \omega_0 > 0$ (see Figure 3.1(b)) for some value of the parameter. It is obvious (and can be rigorously formalized) that we need more parameters to allocate extra eigenvalues on the imaginary axis. Notice that this might